1. a) Suppose that $d_1, d_2, \ldots, d_k$ are the divisors of $m$ and $e_1, e_2, \ldots, e_l$ are the divisors of $n$. Then

$$\sigma(m) = d_1 + d_2 + \ldots + d_k$$
$$\sigma(n) = e_1 + e_2 + \ldots + e_l.$$

If these expressions are multiplied together and the product is expanded with the distributive property, the result is a sum of terms of the form $d_i e_j$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$.

We stated in class that each divisor of $mn$ (when $\gcd(m, n) = 1$) can be written uniquely as $d \cdot e$, where $d \mid m$ and $e \mid n$. Therefore, the $k \cdot l$ products $d_i e_j$ are precisely the divisors of $mn$, with each divisor occurring exactly once. Hence

$$\sigma(m) \sigma(n) = (d_1 + \ldots + d_k)(e_1 + \ldots + e_l)$$
$$= d_1 e_1 + \ldots + d_k e_k + d_k e_l$$
$$= \sigma(mn).$$

b) The divisors of $p^e$ are $1, p, p^2, p^3, \ldots, p^e$. So

$$\sigma(p^e) = 1 + p + p^2 + \ldots + p^e = \frac{p^{e+1} - 1}{p - 1}$$

Ex. 2 $\sigma(10) = \sigma(2) \sigma(5) = (1 + 2)(1 + 5) = 18.$
$$\sigma(20) = \sigma(4) \sigma(5) = (1 + 2 + 4)(1 + 5) = 42.$
$$\sigma(1728) = \sigma(2^6 \cdot 3^3) = \frac{2^7 - 1}{2 - 1} \cdot \frac{3^4 - 1}{3 - 1} = 127 \cdot 40 = 5080.$$
$$\sigma(4100) = \sigma(41 \cdot 2^2 \cdot 5^2) = 42 \cdot (1 + 2 + 4)(1 + 5 + 25) = 42 \cdot 7 \cdot 31 = 9114.$$

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a) Since \( \gcd(2^k, m) = 1 \), it follows that \( \sigma(n) = \sigma(2^k \cdot m) = \sigma(2^k) \sigma(m) = (2^{k+1} - 1) \cdot \sigma(m) \). Thus this must equal \( Z_n \), which is \( 2^{k+1} \cdot m \). Thus
\[
(2^{k+1} - 1) \sigma(m) = 2^{k+1} \cdot m.
\]
Since \( \gcd(2^{k+1} - 1, 2^{k+1}) = 1 \) and \( 2^{k+1} \parallel (2^{k+1} - 1) \cdot \sigma(m) \), it follows that \( 2^{k+1} \mid \sigma(m) \). Let \( l = \frac{\sigma(m)}{2^{k+1}} \); it must be an integer.

Since
\[
\frac{\sigma(m)}{2^{k+1}} = \frac{m}{2^{k+1} - 1},
\]
thus both equal \( l \), and thus
\[
\sigma(m) = l \cdot 2^{k+1}
\]
\&
\[m = l \cdot (2^{k+1} - 1).
\]

b) The numbers \( a \cdot (2^b - 1), a, \) and \( 1 \) are all divisors of \( a \cdot (2^b - 1) \). Since \( a \geq 2 \) and \( 2^b - 1 \geq 2 \), we have the inequalities
\[1 < a < a \cdot (2^b - 1),
\]
hold, so there are distinct divisors. So
\[
\sigma(a(2^b - 1)) = 1 + a + a(2^b - 1) = 1 + a \cdot 2^b
\]
\[\Rightarrow \sigma(a(2^b - 1)) > a \cdot 2^b.
\]

I needed \( a \geq 2 \) so that \( a > 1 \); I needed \( b \geq 2 \) so that \( a < a \cdot (2^b - 1) \).

c) Let \( a = l \) and \( b = k + 1 \). Since \( \sigma(a \cdot (2^b - 1)) = a \cdot 2^b \) (b, part a), the assumptions of part (b) cannot hold: either \( a = l = 1 \) or \( b = l \).
But \( b = k + 1 \geq 2 \) since \( k \geq 1 \) (\( m \) is even). So \( a = 1 \), i.e. \( l = 1 \).
(4) a) \[ a^2 \equiv b^2 \mod p \implies a^2 - b^2 \equiv 0 \mod p \]

\[ \implies (a+b)(a-b) \equiv 0 \mod p \]

\[ \implies p \mid (a+b)(a-b) \text{ (since } p \text{ is prime)} \]

\[ \implies a+b \equiv 0 \mod p \text{ or } a-b \equiv 0 \mod p \]

\[ \implies a \equiv -b \mod p \text{ or } a \equiv b \mod p. \]

b) Suppose for contradiction that \( m \) is prime.

Since \( a^2 \equiv b^2 \mod m \), part (a) implies that either \( a \equiv b \mod m \) or \( a \equiv -b \mod m \). Since \( |a-b| \leq m \) and \( a \neq b \), it cannot be the case that \( a \equiv b \mod m \).

So \( a \equiv -b \mod m \). Then \( m \mid (a+b) \). But \( 2 \leq a+b \leq m \), so the only possibility is \( a=b=\frac{1}{2}m \). But this is a contradiction, since \( a \neq b \).

Therefore \( m \) cannot be prime; it is composite.

c) \[ 150^2 = 22500 \equiv 169 \mod 22331. \]

So \[ 150^2 \equiv 13^2 \mod 22331. \]

This means \( 150^2 - 13^2 \) is divisible by 22331. In fact, it is equal to 22331. But

But \( 1 \leq 13 < 150 \leq \frac{1}{2} \cdot 22331 \). By part (b), it follows that 22331 is composite.

Note: In fact, \[ 22331 = 150^2 - 13^2 = (150+13)(150-13) = 163 \cdot 137. \]