1. Identify the 41 colors with the numbers $0, \ldots, 40$. The strategy is as follows:

- The first gnome studies the colors $C_2, C_3, \ldots, C_{1000}$ of all the rest of the gnomes. He guesses $(C_2 + C_3 + \ldots + C_{1000}) \mod 41$ for his own color. He is probably wrong, but it will be OK! Call his answer $s$.

- Gnome 2 can see $C_3, \ldots, C_{1000}$. She computes $C_3 + \ldots + C_{1000}$ and subtracts it from $s$. She gets $C_2 \mod 41$. So she works out $C_2$ and (correctly) guesses it.

- Gnome 3 knows $C_2$ (since gnome 2 was correct) as well as $C_4, C_5, \ldots, C_{1000}$ (by observation).

- She computes $C_2 + C_4 + \ldots + C_{1000}$, subtracts it from $s$, and obtains a number $\equiv C_3 \mod 41$, which she finds $C_3$ and (correctly) guesses it.

Continuing this way...

- Gnome $n$ knows $s$ as well as $C_2, C_3, \ldots, C_{n-1}$ (by listening) and $C_{n+1}, C_{n+2}, \ldots, C_{1000}$ (by looking).

- She computes $C_2 + C_3 + \ldots + C_{n-1} + C_{n+1} + \ldots + C_{1000}$, subtracts it from $s$, finds the number in $50, \ldots, 403$ congruent to the result, and (correctly) guesses it.

In this way, every gnome but the first is certain to guess correctly.
2) Observe that \( n \equiv -10 \mod (n+10) \).

Hence

\[
(n+10)|((n^3+100) \iff n^3 \equiv -100 \mod (n+10) \\
\iff (-10)^3 \equiv -100 \mod (n+10) \\
\iff 0 \equiv 900 \mod (n+10) \\
\iff (n+10) | 900.
\]

So the largest such \( n \) is \( 900 - 10 = 890 \).

3) a) The congruence \( ax \equiv 1 \mod b \) has a solution \( x \) since \( \gcd(a,b) = 1 \). Let \( u = ax \). Then clearly \( u \equiv 0 \mod a \) and \( u \equiv 1 \mod b \).

Let \( v = 1-u \). Then \( v \equiv 1-0 \equiv 1 \mod a \)
and \( v \equiv 1-1 \equiv 0 \mod b \).

b) Let \( x = c \cdot u + d \cdot v \). Then

\[
x = c \cdot u + d \cdot v \equiv c \mod a
\]
and \( x \equiv c \cdot 0 + d \cdot 1 \equiv d \mod b \),
as desired.

c) We must have \( n = 16 + 17k \) for some \( k \). The number \( k \) must be chosen so that

\[
16 + 17k \equiv 4 \mod 19
\]
\( \iff 17k \equiv -12 \mod 19 \\
\iff -2k \equiv -12 \mod 19 \\
\iff k \equiv 6 \mod 19 \quad (\gcd(-2, 19) = 1). \]
So let
\[ n = 16 + 6 \cdot 197 \]
\[ = 16 + 1182 \]
\[ = 1398. \square \]

(4) We are given that \( (a^{m+1}) | (a^n+1) \), i.e.
\[ a^{n+1} \equiv 0 \mod (a^{m+1}). \]

For convenience let \( M = a^{m+1} \). Observe that \( a^m \equiv -1 \mod M \).
Therefore, writing
\[ n = q \cdot m + r \quad (\text{for } r = n \mod m), \]
it follows that
\[ a^{n+1} \equiv (a^m)^q \cdot a^r + 1 \mod M \]
\[ \equiv (-1)^q \cdot a^r + 1 \mod M \]
\[ \equiv -1 \mod M \]

Now, since \( a^{n+1} \equiv 0 \mod M \), we know that
\[ a^r \equiv (-1)^q \mod M. \]

Since \( 0 \leq r < m \), we know that \( 1 \leq a^r \leq a^{m-1} \) and \( a^{m-1} \leq a^{m-2} \) since therefore \( m \geq 2 \); we must have that either \( a^r = 1 \) or \( a^r = a^{m-1} \).

In the first case, \( r = 0 \), so \( m \mid n \) as desired. In the second
case, \( r \) must also be 0 or else \( a \) divides both \( a \) and \( a^r \).

hence \( a \mid 1 \), which is impossible. So in either case, \( r = 0 \) and \( m \mid n \), as desired.

(5) 43 is prime, so by Fermat's Little Theorem,

\[
19^{5085} \equiv 19^{5085 \mod 43} \mod 43
\]

Now, \( 5085 = 121 \cdot 42 + 3 \), so

\[
19^{5085} = 19^3 \mod 43.
\]

\[
= (19^2) \cdot 19 = 361 \cdot 19 \mod 43
\]

\[
= 17 \cdot 19 = 323
\]

\[
= 22 \mod 43
\]

(6) We wish to find an exponent \( p \) st. \((x^{17})^p \equiv x \mod 43\).

It suffices to solve

\[
17f \equiv 1 \mod 42.
\]

Using the extended Euclidean algorithm:

\[
\begin{align*}
[8] &= (42) - 2 \cdot (17) \\
[1] &= (17) - 2 \cdot [8] \\
&= (17) - 2 \cdot (42) + 4 \cdot (17) \\
&= 5(17) - 2(42)
\end{align*}
\]

Hence \( 25 = 1 \mod 42 \).

\[
egin{align*}
5^3 &= 125 \equiv 17 \\
5^4 &= 20
\end{align*}
\]

\[
5^5 = -100 \equiv 29 \quad \text{So} \quad x \equiv 29 \mod 43
\]