1. One thousand gnomes have been imprisoned by an evil wizard. The wizard informs the gnomes that the following day they will line up, facing forward, and that he will place a hat on each gnome. Each hat will be one of 41 possible colors. Starting from the back of the line, each gnome will then be permitted to guess their own hat color. After all the gnomes have finished guessing, the gnomes that guessed correctly will be freed. Determine a strategy for the gnomes guaranteeing that at most one of them will not be freed. Each gnome can see all of the hats in front of them, and hear all of the guesses made by gnomes behind them, but they cannot see their own hat color.[1]

2. Determine the largest integer n such that $n + 10$ divides $n^3 + 100$.

3. Suppose that a, b are two positive integers such that $\gcd(a, b) = 1$.

   (a) Prove that there exists integers u, v such that the following congruences hold.

   $$u \equiv 0 \pmod{a} \quad u \equiv 1 \pmod{b}$$
   $$v \equiv 1 \pmod{a} \quad v \equiv 0 \pmod{b}$$

   (b) Prove that for any two integers c, d, there exists an integer x such that $x \equiv c \pmod{a}$ and $x \equiv d \pmod{b}$.

   (c) Find an integer n such that $n \equiv 16 \pmod{17}$ and $n \equiv 4 \pmod{19}$.

4. Let a, m, n be positive integers, with $a \geq 2$. Prove that if $a^m + 1$ divides $a^n + 1$, then m divides n.

5. Determine the remainder when 19^{5085} is divided by 43.

6. Solve the congruence $x^{17} \equiv 5 \pmod{43}$.

[1] Don't worry: once these 999 gnomes are free they will be powerful enough to come back and rescue the last one.