1. For each number \( k \) between 0 and 8 (inclusive), either find two different prime numbers \( p \) such that \( p \equiv k \pmod{9} \), or prove that it is impossible to do so.

   \textit{Aside.} There is a celebrated theorem of Dirichlet that says (in part) that for each of these \( k \) there are in fact infinitely many such primes. We may prove some special cases of this later.

2. Find all incongruent solutions to the following congruences.
   
   (a) \( 55x \equiv 30 \pmod{625} \)
   (b) \( 55x \equiv 30 \pmod{1331} \)

3. Two integers \( x \) and \( y \) are called \textit{inverses} modulo 24 if \( xy \equiv 1 \pmod{24} \).

   (a) Prove that if \( x \) has an inverse modulo 24, then this inverse is unique modulo 24 (that is, if \( y_1, y_2 \) are both inverses of \( x \) modulo 24, then \( y_1 \equiv y_2 \pmod{24} \)).
   (b) Which \( x \) between 0 and 23 inclusive have inverses modulo 24? For each such \( x \), find an inverse \( y \).

4. A notion closely related to the greatest common divisor \( \gcd(a, b) \) is the least common multiple \( \lcm(a, b) \), which is defined to be the smallest positive integer that is a multiple of both \( a \) and \( b \). Throughout this problem, assume that \( a, b \) are positive integers (in particular, that neither is zero).

   (a) Prove that if \( \gcd(a, b) = 1 \), then \( \lcm(a, b) = ab \).
   (b) Prove that for any positive integer \( k \), \( \lcm(ka, kb) = k \cdot \lcm(a, b) \).
   (c) Deduce from parts (a) and (b) that \( \gcd(a, b) \cdot \lcm(a, b) = ab \).

5. Find three positive integers \( a, b, c \) with no common factors such that \( a^2 + b^2 = c^4 \).