1. For each number $k$ between 0 and 8 (inclusive), either find two different prime numbers $p$ such that $p \equiv k \pmod 9$, or prove that it is impossible to do so.

   *Aside.* There is a celebrated theorem of Dirichlet that says (in part) that for each of these $k$ there are in fact infinitely many such primes. We may prove some special cases of this later.

2. Find all incongruent solutions to the following congruences.

   (a) $55x \equiv 30 \pmod{625}$

   (b) $55x \equiv 30 \pmod{1331}$

3. Two integers $x$ and $y$ are called *inverses* modulo 24 if $xy \equiv 1 \pmod{24}$.

   (a) Prove that if $x$ has an inverse modulo 24, then this inverse is unique modulo 24 (that is, if $y_1, y_2$ are both inverses of $x$ modulo 24, then $y_1 \equiv y_2 \pmod{24}$).

   (b) Which $x$ between 0 and 23 inclusive have inverses modulo 24? For each such $x$, find an inverse $y$.

4. A notion closely related to the greatest common divisor $\gcd(a, b)$ is the least common multiple $\operatorname{lcm}(a, b)$, which is defined to be the smallest positive integer that is a multiple of both $a$ and $b$. Throughout this problem, assume that $a, b$ are positive integers (in particular, that neither is zero).

   (a) Prove that if $\gcd(a, b) = 1$, then $\operatorname{lcm}(a, b) = ab$.

   (b) Prove that for any positive integer $k$, $\operatorname{lcm}(ka, kb) = k \cdot \operatorname{lcm}(a, b)$.

   (c) Deduce from parts $(a)$ and $(b)$ that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.

5. Find three positive integers $a, b, c$ with no common factors such that $a^2 + b^2 = c^4$.