Math 42
Spring 2015

PSet 3 Solutions

(1) We are considering the equation $5x+2y=79$, where x,y should be nonnegative integers. One solution is

$$(5)\cdot 15 + (2)\cdot 2 = 79.$$

Any other solution must satisfy $(5)\cdot(x-15)+(2)\cdot(y-2)=0$,

here we use that $\gcd(5,2)=1$.

which implies that $2 \mid 5(x-15)$, which implies $2 \mid (x-15)$. Similarly, $5 \mid (y-2)$. So any solution must have the form

$$(5)\cdot(15-2k) + (2)\cdot(2+5k) = 79.$$

Now, k must be ≥ 0 or else $2+5k$ is negative. Also, $k \leq 7$ or else $15-2k$ is negative. So $k=0,1,2,\cdots,7$ all give solutions, and these are all solutions. Therefore there are eight solutions total, namely:

$$\underset{k=0}{(15,2)},\ \underset{k=1}{(13,7)},\ \underset{k=2}{(11,12)},\ \underset{k=3}{(9,17)},\ \underset{k=4}{(7,22)},$$

$$\underset{k=5}{(5,27)},\ \underset{k=6}{(3,32)},\ \text{and}\ \underset{k=7}{(1,37)}.$$

(2) a) $(20)-6(3)=2$ and $3-2=1$, hence $(3)-[(20)-6(3)]=1$, i.e. $(3)\cdot 7-(20)\cdot 1=1$ $(w=7,\ z=-1)$.

b) Apply the Euclidean algorithm to 6 and 15 to get $(15)\cdot 1-(6)\cdot 2=3$. Multiplying by w gives

$$(15)\cdot 7-(6)\cdot 14 = 21. \quad (x=-14,\ y=7)$$

c) $(x,y,z) = \underline{(-14, 7, 1)}$ gives

$$-(6)\cdot 14 + (15)\cdot 7 - (20)\cdot 1 = 1.$$

d) First, applying the euclidean algorithm to 155 & 341:

$$
\begin{array}{lll}
341 & 155 & \\
155 & 31 & = \underline{341 - 2\cdot 155} \\
31 & 0 & = 155 - 5\cdot 31
\end{array}
$$

so $\gcd(155, 341) = 31$. Hence we must solve $31w + 385z = 1$, then set $x = -2w$ and $y = w$ to guarantee that $155x + 341y = 31w$. Using the Euclidean algorithm:

$$
\begin{array}{lll}
385 & 31 & \\
31 & 13 & = (385) - 12\cdot(31) \\
13 & 5 & = (31) - 2(13) = (31) - 2\cdot[(385) - 12(31)] \\
 & & = -2(385) + 25\cdot(31) \\
5 & 3 & = (13) - 2(5) = [(385) - 12(31)] - 2[-2(385) + 25(31)] \\
 & & = 5\cdot(385) - 62\cdot(31) \\
3 & 2 & = (5) - (3) = [-2(385) + 25(31)] - [5(385) - 62(31)] \\
 & & = -7\cdot(385) + 87\cdot(31) \\
2 & 1 & = (3) - (2) \\
 & & = [5(385) - 62(31)] - [-7(385) + 87(31)] = 12(385) - 149(31) \\
1 & 0 &
\end{array}
$$

so

$$-149\cdot(31) + 12(385) = 1$$

$$\Rightarrow -149\cdot[(341) - 2\cdot(155)] + 12(385) = 1$$

$$298\cdot(155) - 149\cdot(341) + 12\cdot(385) = 1$$

ie. $(x,y,z) = (298, -149, 12)$ is one solution (there are many others).

③ Since $\gcd(a,b)=1$, there exist $x,y\in\mathbb{Z}$ such that

$$ax + by = 1.$$

Multiplying this equation by c:

$$acx + bcy = c$$

$$ab\cdot\left[\frac{c}{b}\cdot x + \frac{c}{a}\cdot y\right] = c$$

Now, since $b|c$ and $a|c$, the expression in square brackets is an integer. Therefore $ab|c$ (c is an integer multiple of ab).

④ a) $n=3$: $F_3=2$ and $F_4=3$ so $-F_3+F_4=1$.
$n=4$: $F_4=3$ and $F_5=5$ so $2\cdot F_4 - F_5 = 1$.
$n=5$: $F_5=5$ and $F_6=8$ so $-3\cdot F_5 + 2\cdot F_6 = 1$.
$n=6$: $F_6=8$ and $F_7=13$. This is getting harder to guess, so I'll use the Euclidean algorithm:

13   8

8    $5 = (13) - (8)$

5    $3 = (8) - (5) = (8) - [(13) - (8)]$

$= 2\cdot(8) - (13)$

3    $2 = (5) - (3) = [(13) - (8)] - [2\cdot(8) - (13)]$

$= 2(13) - 3(8)$

2    $1 = (3) - (2) = [2(8) - (13)] - [2(13) - 3(8)]$

$= 5(8) - 3(13).$

So $5\cdot F_6 - 3\cdot F_7 = 1.$

$n=7$: This time I'll economize a bit. The first step of the Euclidean algorithm is (since $F_7 = 13$ $F_8 = 21$)

21   13

13    $8 = (21) - (13)$

but I already know that $1 = 5\cdot(8) - 3(13)$;

so $1 = 5\cdot[(21) - (13)] - 3(13) = 5(21) - 8\cdot(13)$,

ie. $-8F_7 + 5F_8 = 1.$

$n=8$: $F_8 = 21$ $F_9 = 34$. Going off the idea from last time, $(13) = (34) - (21)$ and

$1 = 5(21) - 8(33) = 5(21) - 8[(34) - (21)]$

$= 13\cdot(21) - 8\cdot(34)$

so $13\cdot F_8 - 8\cdot F_9 = 1.$

$n=9$: $F_9 = 34$ $F_{10} = 55$. $(21) = (55) - (34)$ and

$1 = 13(21) - 8(34)$, so $1 = 13[(55) - (34)] - 8(34)$

$= -21(34) + 13(55)$

so $-21\cdot F_9 + 13\cdot F_{10} = 0.$

Summarizing these results:

n	3	4	5	6	7	8	9
x_n	-1	2	-3	5	-8	13	-21
y_n	1	-1	2	-3	5	-8	13

Indeed, there is a pattern: each x_n & y_n is itself plus or minus a Fibonacci number, namely:

$$x_n = (-1)^n F_{n-1}$$
$$y_n = (-1)^{n+1} F_{n-2}$$

and we have essentially found the identity

$$F_{n-1} \cdot F_n - F_{n-2} \cdot F_{n+1} = (-1)^n,$$

which could also be shown by induction on n.

b) We proceed by induction. The base case is $n=0$. In this case $\gcd(F_n, F_{n+1}) = \gcd(F_0, F_1) = \gcd(0,1) = 1$.

For the inductive step, assume $\gcd(F_{n-1}, F_n) = 1$.
Now, since we've shown that $\gcd(a,b) = \gcd(a, b-a)$:

$$\begin{aligned} \gcd(F_n, F_{n+1}) &= \gcd(F_n, F_n + F_{n-1}) \\ &= \gcd(F_n, F_n + F_{n-1} - F_n) \\ &= \gcd(F_n, F_{n-1}). \end{aligned}$$

By the inductive hypothesis, this last line is 1. Hence $\gcd(F_n, F_{n+1}) = 1$ as well, completing the induction.

(5) a) Try all residues:

x	0	1	2	3	4
$7x \% 5$	0	2	4	1	3

so the solution is $\boxed{x \equiv 2 \bmod 5}$

b)

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$x^2 \% 13$	0	1	4	9	3	12	10	10	12	3	9	4	1

so the solutions are $\boxed{x \equiv 4 \bmod 13 \ \& \ x \equiv 9 \bmod 13}$
(equivalently, $x \equiv \pm 4 \bmod 13$).

c)

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$9x \% 15$	0	9	3	12	6	0	9	3	12	6	0	9	3	12	6

There are three solutions:

$$\boxed{\begin{array}{l} x \equiv 4 \bmod 15 \\ x \equiv 9 \bmod 15 \\ x \equiv 14 \bmod 15 \end{array}}$$

<u>Note</u> you could also express all three quite efficiently by writing $x \equiv 4 \bmod 5$.