1. We are considering the equation $5x + 2y = 79$, where $x, y$ should be nonnegative integers. One solution is

$$(5) \cdot 15 + (2) \cdot 2 = 79.$$ 

Any other solution must satisfy $(5)(x-15) + (2)(y-2) = 0$, which implies that $2 \mid 5(x-15)$, which implies $2 \mid (x-15)$. Similarly, $5 \mid (y-2)$. So any solution must have the form

$$(5) \cdot (15-2k) + (2) \cdot (2+5k) = 79.$$ 

Now, $k$ must be $\geq 0$ or else $2+5k$ is negative. Also, $k \leq 7$ or else $15-2k$ is negative. So $k = 0, 1, 2, \ldots, 7$ all give solutions, and there are all solutions. Therefore there are eight solutions total, namely:

$k = 0, 1, 2, 3, 4, 5, 6, 7$

$$(15, 2), (13, 7), (11, 12), (9, 17), (7, 22), (5, 27), (3, 32), \text{ and } (1, 37).$$

2. a) $(20) - 6(3) = 2$ and $3 - 2 = 1$, hence $(3) - [(20) - 6(3)] = 1$, i.e. $(3) \cdot 7 - (20) \cdot 1 = 1$ ($w = 7, z = -1$).

b) Apply the Euclidean algorithm to 6 and 15 to get $(15) \cdot 1 - (6) \cdot 2 = 3$. Multiplying by $w$ gives

$$(15) \cdot 7 - (6) \cdot 14 = 21. \quad (x = -14, \ y = 7)$$
c) \((x,y,z) = (-14,7,1)\) gives

\[-(6) \cdot 14 + (15) \cdot 7 - (20) \cdot 1 = 1.\]

d) First, applying the euclidean algorithm to 155 & 341:

\[
\begin{array}{c}
341 \\
\underline{155} \\
155 \div 31 = 341 - 2 \cdot 155 \\
\underline{31} \\
0 = 155 - 5 \cdot 31
\end{array}
\]

so \(\gcd(155, 341) = 31\). Hence we must solve \(31w + 385z = 1\), then set \(x = -2w\) and \(y = w\) to guarantee that \(155x + 341y = 31w\). Using the Euclidean algorithm:

\[
\begin{array}{c}
385 \\
\underline{31} \\
31 \div 13 = (385) - 12 \cdot (31) \\
\underline{13} \\
5 \div 3 = (31) - 2 \cdot (13) = (31) - 2 \cdot \left[ (385) - 12 \cdot (31) \right] \\
\underline{5} \\
4 \div 3 = (13) - 2 \cdot (5) = \left[ (385) - 12 \cdot (31) \right] - 2 \left[ (385) - 25 \cdot (31) \right] \\
\underline{3} \\
2 \div 1 = (5) - (3) = \left[ -2 \cdot (385) + 25 \cdot (31) \right] - \left[ 5 \cdot (385) - 62 \cdot (31) \right] \\
\underline{2} \\
1 \div 0 = (3) - (2) = \left[ 5 \cdot (385) - 62 \cdot (31) \right] - \left[ -7 \cdot (385) + 87 \cdot (31) \right] = 12 \cdot (385) - 149 \cdot (31)
\end{array}
\]
\(-149 \cdot (31) + 12 \cdot (385) = 1\)

\[
\Rightarrow -149 \left[ (341) - 2 \cdot (155) \right] + 12 \cdot (385) = 1
\]

\[
298 \cdot (155) - 149 \cdot (341) + 12 \cdot (385) = 1
\]

ie. \((x,y,z) = (298, -149, 12)\) is one solution (there are many others).

(3) Since \(\gcd(a,b) = 1\), there exist \(x,y \in \mathbb{Z}\) such that

\[
ax + by = 1.
\]

Multiplying this equation by \(c\):

\[
ac x + bc y = c
\]

\[
ab \cdot \left[ \frac{c}{b} \cdot x + \frac{c}{a} \cdot y \right] = c
\]

Now, since \(b/c\) and \(a/c\), the expression in square brackets is an integer. Therefore \(ab/c\) (\(c\) is an integer multiple of \(ab\)).

(4) a) \(n=3\): \(F_3 = 2\) and \(F_4 = 3\) so \(-F_3 + F_4 = 1\).

\(n=4\): \(F_4 = 3\) and \(F_5 = 5\) so \(2 \cdot F_4 - F_5 = 1\).

\(n=5\): \(F_5 = 5\) and \(F_6 = 8\) so \(-3 \cdot F_5 + 2 \cdot F_6 = 1\).

\(n=6\): \(F_6 = 8\) and \(F_7 = 13\). This is getting harder to guess, so I'll use the Euclidean algorithm.
\[
5 = (13) - (8)
\]
\[
3 = (8) - (5) = (8) - [(13) - (8)] = 2 \cdot (8) - (13)
\]
\[
2 = (5) - (3) = [(13) - (8)] - [2 \cdot (8) - (13)] = 2(13) - 3(8)
\]
\[
1 = (3) - (2) = [2(8) - (13)] - [2(13) - 3(8)] = 5(8) - 3(13).
\]
So 
\[5 \cdot F_6 - 3 \cdot F_7 = 1.\]

\[n = 7; \] This time I'll economize a bit. The first step of the Euclidean algorithm is 
\[(\text{since } F_7 = 13, F_8 = 21)\]
\[
21 = 13
\]
\[
13 = (21) - (13)
\]
but I already know that 
\[1 = 5 \cdot (8) - 3(13);\]
so 
\[1 = 5 \cdot [(21) - (13)] - 3(13) = 5(21) - 8 \cdot (13),\]
i.e. 
\[-8F_7 + 5F_8 = 1.\]

\[n = 8; \] \(F_8 = 21, F_9 = 34. \) Going off the idea from last time,
\[(13) = (34) - (21)\] and 
\[1 = 5(21) - 8(34) = 5(21) - 8[(34) - (21)]\]
\[= 13 \cdot (21) - 8 \cdot (34)\]
so 
\[13 \cdot F_8 - 8 \cdot F_9 = 1.\]

\[n = 9; \] \(F_9 = 34, F_{10} = 55. \) \((21) = (55) - (34)\) and 
\[1 = 13(21) - 8(34), \] so 
\[1 = 13[(55) - (34)] - 8(34)\]
\[= -21(34) + 13(55)\]
so 
\[-21 \cdot F_9 + 13 \cdot F_{10} = 0.\]
Summarizing these results:

<table>
<thead>
<tr>
<th>n</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>xn</td>
<td>-1</td>
<td>2</td>
<td>-3</td>
<td>5</td>
<td>-8</td>
<td>13</td>
<td>-21</td>
</tr>
<tr>
<td>yn</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-3</td>
<td>5</td>
<td>-8</td>
<td>13</td>
</tr>
</tbody>
</table>

Indeed, there is a pattern: each $x_n$ & $y_n$ is itself plus or minus a Fibonacci number, namely:

$$x_n = (-1)^n F_{n-1}$$
$$y_n = (-1)^{n+1} F_{n-2}$$

and we have essentially found the identity

$$F_{n-1} \cdot F_n - F_{n-2} \cdot F_{n+1} = (-1)^n,$$

which could also be shown by induction on $n$.

b) We proceed by induction. The base case is $n=0$. In this case, $gcd(F_0, F_{n+1}) = gcd(F_0, F_1) = gcd(0, 1) = 1$.

For the inductive step, assume $gcd(F_{n-1}, F_n) = 1$.
Now, since we've shown that $gcd(a, b) = gcd(a, b-a)$:

$$gcd(F_n, F_{n+1}) = gcd(F_n, F_{n}+F_{n-1})$$
$$= gcd(F_n, F_{n}+F_{n-1}-F_n)$$
$$= gcd(F_n, F_{n-1}).$$

By the inductive hypothesis, this last line is 1. Hence $gcd(F_n, F_{n+1}) = 1$ as well, completing the induction.
a) Try all residues:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 \mod 5</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

so the solution is \( x \equiv 2 \mod 5 \)

b) 

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 \mod 13 )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>3</td>
<td>12</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>3</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

so the solutions are \( x \equiv 4 \mod 13 \) & \( x \equiv 9 \mod 13 \) (equivalently, \( x \equiv \pm 4 \mod 13 \)).

c) 

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 9x \mod 15 )</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>12</td>
<td>6</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>12</td>
<td>6</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>

There are three solutions: \( x \equiv 4 \mod 15 \), \( x \equiv 9 \mod 15 \), \( x \equiv 14 \mod 15 \)

Note you could also express all three quite efficiently by writing \( x \equiv 4 \mod 5 \).