P. Set 1 Solutions

Math 42
Spring 2015

(1) $S(1) = 1$

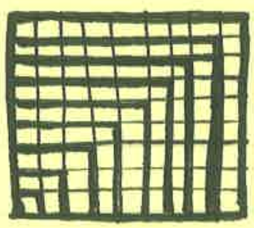
$S(2) = 1+3=4$

$S(3) = 1+3+5=9$

$S(4) = 1+3+5+7=16$

$S(5) = 1+3+5+7+9=25$

The pattern is that $S(n)=n^2$ for all $n \in \mathbb{N}$. One way to see this visually is to divide an n by n box of squares into a sequence of L-shaped parts, for example:



This 9×9 box is split into L's of size 1, 3, 5, 7, 9, 11, 13, 15, and 17.

(2) These can be found by some guesswork, as follows: for various values of a, find the largest b such that $2b^2 < a^2$, and compute a^2-2b^2 to see if it is 1. You can also save some effort by noticing that a must be odd, otherwise a^2-2b^2 would be even.

a	largest b	a^2-2b^2
3	2	$9-2\cdot 4=1$
5	3	$25-2\cdot 9=7$
7	4	$49-2\cdot 16=15$
9	6	$81-2\cdot 36=9$
11	7	$121-2\cdot 49=23$
13	9	$169-2\cdot 81=7$
15	11	$225-2\cdot 121=3$
17	12	$289-2\cdot 144=1$

This search yields the first two examples:

$\boxed{(3,2) \text{ and } (17,12)}$

Note. This equation is an example of "Pell's equation," which is discussed in chapter 32. There are several efficient ways to enumerate its solutions. One way to view the solutions is that (a,b) is a solution if a/b is a good rational approximation of the irrational number $\sqrt{2}$. For example, $17/12 \approx 1.4167$, while $\sqrt{2} \approx 1.4142$.

(3) Below, I have arranged the numbers 1–120 in three columns, and identified the primes using the Sieve of Eratosthenes.

~~1~~	2	3	1	31	~~32~~	~~33~~	2	61	~~62~~	~~63~~	3	~~91~~	~~92~~	~~93~~	
~~4~~	5	~~6~~	2	~~34~~	~~35~~	~~36~~		~~64~~	~~65~~	~~66~~		~~94~~	~~95~~	~~96~~	
7	~~8~~	~~9~~	1	37	~~38~~	~~39~~	1	67	~~68~~	~~69~~	2	97	~~98~~	~~99~~	2
~~10~~	11	~~12~~	2	~~40~~	41	~~42~~	2	~~70~~	71	~~72~~	3	~~100~~	101	~~102~~	3
13	~~14~~	~~15~~	1	43	~~44~~	~~45~~	1	73	~~74~~	~~75~~	2	103	~~104~~	~~105~~	2
~~16~~	17	~~18~~	2	~~46~~	47	~~48~~	2	~~76~~	~~77~~	~~78~~		~~106~~	107	~~108~~	3
19	~~20~~	~~21~~	1	~~49~~	~~50~~	~~51~~		79	~~80~~	~~81~~	1	109	~~110~~	~~111~~	2
~~22~~	23	~~24~~	2	~~52~~	53	~~54~~	3	~~82~~	83	~~84~~	2	~~112~~	113	~~114~~	3
~~25~~	~~26~~	~~27~~		~~55~~	~~56~~	~~57~~		~~85~~	~~86~~	~~87~~		~~115~~	~~116~~	~~117~~	
~~28~~	29	~~30~~	3	~~58~~	59	~~60~~	4	~~88~~	89	~~90~~	3	~~118~~	~~119~~	~~120~~	

Counting primes in each column gives the score:

Team 1: 13
Team 2: 16

Interestingly, you may observe that team 2 is always ~~tie~~ in the lead during these 120 rounds (I've written the size of their lead in pencil), but never by more than 4. You might guess that this is true for all time, but in fact it is not. Remarkably, Team 1 will take the lead for the first time only in round 608,981,813,029. They will later lose the lead again.

(4) ↯ Suppose that $\sqrt{3} = \mathbb{Q}$.

Then there are $p, q \in \mathbb{N}$ <u>with no common factors</u> such that $\sqrt{3} = p/q$, i.e. $3q^2 = p^2$.

Since 3 divides $3q^2$, 3 divides p^2. This implies that <u>3 divides</u> p. [You may assume that if 3 divides the square of a number, then it divides the original number]

Writing $p = 3 \cdot r$ (for some $r \in \mathbb{N}$), it follows that $3q^2 = 9r^2$, or $q^2 = 3r^2$. Therefore 3 divides q^2. This implies that <u>3 divides q</u>.

But since 3 divides both p and q, they have a common factor after all, which is a contradiction. ↯. So $\underline{\underline{\sqrt{3} \notin \mathbb{Q}}}$.

↯ Suppose that $\sqrt[3]{2} \in \mathbb{Q}$.

Then there are $p, q \in \mathbb{N}$ with no common factors such that $p/q = \sqrt[3]{2}$, i.e. $2p^3 = q^3$.

Since $2p^3$ is even, q^3 is even, hence <u>q is even</u>.

So $q = 2 \cdot r$ for some $r \in \mathbb{N}$.

Thus $2p^3 = (2r)^3 = 8r^3$, i.e. $p^3 = 4r^3$.

Therefore p^3 is even, so $\underline{p \text{ is even}}$.

But this means that both p and q are even, which is a contradiction. ↯ So $\sqrt[3]{2} \notin \mathbb{Q}$.

5 For both problems, use the following recipe for PPTs:

$$a = st \quad b = \frac{s^2-t^2}{2} \qquad c = \frac{s^2+t^2}{2}$$

where s,t are odd, $s>t$, and s,t have no common factors.

a) Either $s=33$, $t=1$ or $s=11$, $t=3$ will give $a=33$. These give the following two PPTs:

$$\left(33\cdot 1, \frac{33^2-1}{2}, \frac{33^2+1}{2}\right) = \underline{(33, 544, 545)}$$

$$\left(11\cdot 3, \frac{11^2-3^2}{2}, \frac{11^2+3^2}{2}\right) = \underline{(33, 56, 65)}$$

(either one will be accepted).

b) We want s,t so that $\frac{s^2+t^2}{2} = 85$, i.e. $s^2+t^2 = 170$. There are two choices:

$s=13, t=1$ gives $\left(13\cdot 1, \frac{13^2-1}{2}, \frac{13^2+1}{2}\right) = \underline{(13, 84, 85)}$

$s=11, t=7$ gives $\left(11\cdot 7, \frac{11^2-7^2}{2}, \frac{11^2+7^2}{2}\right) = \underline{(77, 36, 85)}$

(either will be accepted).

c) Both (a) and (b) have two possible answers shown above.