1. By Fermat's Little Theorem, \( g^{p-1} \equiv 1 \mod p \). Therefore, if \( x = (p-1)q + r \), then

\[
g^x = g^{(p-1)q + r} = (g^{p-1})^q \cdot g^r = \equiv g^r \mod p.
\]

In other words, \( g^x \equiv g^{x \mod (p-1)} \mod p \), where \( x \mod (p-1) \) denotes the remainder when \( x \) is divided by \( (p-1) \).

Since \( g \) is a primitive root, the numbers \( g^0, g^1, \ldots, g^{p-2} \) are all distinct modulo \( p \). So we know that \( x \equiv y \mod (p-1) \).

Therefore:

\[
g^x \equiv g^y \mod p \iff \text{true if and only if } g^{x \mod (p-1)} \equiv g^{y \mod (p-1)}
\]

which is true if and only if \( x \mod (p-1) = y \mod (p-1) \).

Then we know:

\[
(18^x)^2 \cdot 18^y \equiv 18^{91} \mod 101
\]

\[
(18^x)^2 \cdot 18^y \equiv 18^{91} \mod 101
\]

\[
(=) \quad 18^{2x+y} \equiv 18^{91} \mod 101
\]

\[
(=) \quad 2x+y \equiv 91 \mod 100 \quad (1)
\]

(by problem 1, since 18 is a primitive root).

By similar logic, \( x+2y \equiv 13 \mod 100 \). \( \quad (2) \)
Therefore:
\[ y \equiv 91 - 2x \mod 100 \quad \text{(from (1))} \quad (3) \]
\[ \Rightarrow \quad x + 2(91 - 2x) \equiv 13 \mod 100 \quad \text{(from (2))} \]
\[ \Rightarrow \quad -3x + 182 \equiv 13 \mod 100 \]
\[ \Rightarrow \quad -69 \equiv 3x \mod 100 \]
\[ \Rightarrow \quad \frac{69}{3} \equiv x \mod 100 \quad \text{(since \ \gcd(3, 100) = 1)} \]
so \[ x \equiv 23 \mod 100 \]

Thus using (3),
\[ y \equiv 91 - 2 \cdot 23 \mod 100 \]
\[ y \equiv 45 \mod 100. \]

Therefore (modulo 100), \( x \) must be 23 & \( y \) must be 45.

\( \square \)

(3) a) \[ 4370 = 2 \cdot 5 \cdot 437 \]
\[ = 2 \cdot 5 \cdot 19 \cdot 23 \]

since 23 \( \equiv 3 \mod 4 \) and appear once in the prime factorization, 4370 \( \neq \) not a sum of two squares.

b) \[ 1885 = 5 \cdot 377 \]
\[ = 5 \cdot 13 \cdot 29. \]

All three of these primes are sos.

\[ 5 = 2^2 + 1^2 \]
\[ 13 = 3^2 + 2^2 \]
\[ 29 = 5^2 + 2^2 \]
we can combine as follows:

\[ 5 \cdot 13 = (2 \cdot 3 + 1 \cdot 2)^2 + (2 \cdot 2 - 1 \cdot 3)^2 \]

\[ = 8^2 + 1^2 \]

\[ (5 \cdot 13) \cdot 29 = (8 \cdot 5 + 1 \cdot 2)^2 + (8 \cdot 2 - 5 \cdot 1)^2 \]

\[ = 42^2 + 11^2 \]

There are three other possible answers: (you only need to give one).

\[
\begin{align*}
5 \cdot 13 &= (2 \cdot 3 - 1 \cdot 2)^2 + (2 \cdot 2 + 1 \cdot 3)^2 \\
&= 4^2 + 7^2 \\
(5 \cdot 13) \cdot 29 &= (4 \cdot 5 + 7 \cdot 2)^2 + (4 \cdot 2 - 5 \cdot 7)^2 \\
&= 34^2 + (-27)^2 \\
&= 34^2 + 27^2
\end{align*}
\]

\[
\begin{align*}
5 \cdot 13 &= (2 \cdot 3 + 1 \cdot 2)^2 + (2 \cdot 2 - 1 \cdot 3)^2 \\
&= 4^2 + 7^2 \\
(5 \cdot 13) \cdot 29 &= (4 \cdot 5 - 7 \cdot 2)^2 + (4 \cdot 2 + 5 \cdot 7)^2 \\
&= 6^2 + 43^2 \\
&= 6^2 + 43^2
\end{align*}
\]

\[
\begin{align*}
5 \cdot 13 &= 8^2 + 1^2 \\
&= 8^2 + 1^2 \\
(5 \cdot 13) \cdot 29 &= (8 \cdot 5 - 1 \cdot 2)^2 + (8 \cdot 2 + 5 \cdot 1)^2 \\
&= 38^2 + 21^2
\end{align*}
\]

c) 1189 = 29.41. Both primes are 1 mod 4, so they are SOTS.

\[
\begin{align*}
29 &= 5^2 + 2^2 \\
41 &= 5^2 + 4^2 \\
\Rightarrow 29.41 &= (5.5 + 2.4)^2 + (5.4 - 2.5)^2 \\
&= 33^2 + 10^2
\end{align*}
\]

The other possible solution is

\[
\begin{align*}
29.41 &= (5.5 - 2.4)^2 + (5.4 + 2.5)^2 \\
&= 17^2 + 30^2
\end{align*}
\]
d) \[ 3185 = 5 \cdot 637 \]

\[ = 5 \cdot 7^2 \cdot 13. \]

5 and 13 are \( \equiv 1 \) mod 7, and 7 occurs an even number of times, so 3185 is SOTS.

\[ 5 = 2^2 + 1^2 \]
\[ 13 = 2^2 + 3^2 \]

\[ = 5 \cdot 13 = (2 \cdot 2 + 1 \cdot 3)^2 + (2 \cdot 3 - 1 \cdot 2)^2 = 7^2 + 4^2 \]

(or \[ = (2 \cdot 2 - 1 \cdot 3)^2 + (2 \cdot 3 + 1 \cdot 2)^2 = 8^2 + 1^2 \].)

The factor of 7 can be introduced only by multiplying both terms to be squared by 7.

\[ (5 \cdot 13) \cdot 7^2 = (7 \cdot 7)^2 + (4 \cdot 7)^2 \]
\[ = \boxed{49^2 + 28^2} \]

The other possible solution is \( (8 \cdot 7)^2 + (1 \cdot 7)^2 \)
\[ = \boxed{56^2 + 7^2} \]

4) \[ 557^2 + 55^2 = 26 \cdot 12049 \]

\[ 557 \equiv 11 \, \text{mod} \, 26 \] and \[ 55 \equiv 3 \, \text{mod} \, 26 \], so
decrease to:

\[ \left( \frac{557 \cdot 11 + 55 \cdot 3}{26} \right)^2 + \left( \frac{557 \cdot 3 - 55 \cdot 11}{26} \right)^2 = \frac{11^2 + 3^2}{26} \cdot 12049 \]
\[ \left( \frac{6292}{26} \right)^2 + \left( \frac{1066}{26} \right)^2 = \frac{130}{26} \cdot 12049 \]
\[ 242^2 + 41^2 = 5 \cdot 12049 \]
descend again:

\[ 242 \equiv 2 \mod 5 \quad 41 \equiv 1 \mod 5 \]

so go to

\[
\left( \frac{242 \cdot 2 + 41 \cdot 1}{5} \right)^2 + \left( \frac{242 \cdot 1 - 41 \cdot 2}{5} \right)^2 = \frac{2^2 + 1^2}{5} \cdot 12049
\]

\[
\left( \frac{525}{5} \right)^2 + \left( \frac{180}{5} \right)^2 = 17049
\]

\[ 105^2 + 32^2 = 12049 \]