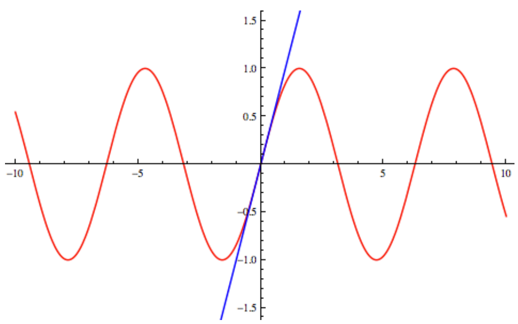
Taylor approximation

5 October 2011

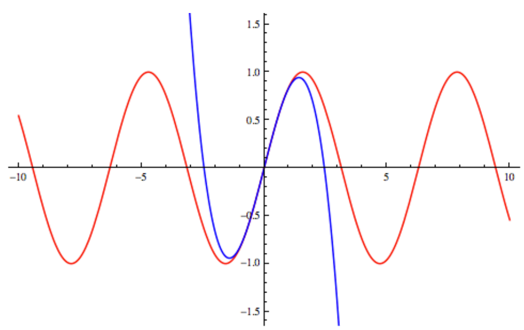
1. The derivatives of the function $f(x) = \sin x$ at $x = 0$ follow the pattern: $1, 0, -1, 0, 1, 0, -1, \ldots$ (i.e. $f'(0) = 1$, $f''(0) = 0$, etc.). The following pictures show polynomials that have been rigged specifically to match these derivatives at 0. Determine which polynomial is shown in each picture.

   (a) This polynomial has $p(0) = 0,\ p'(0) = 1$, and all other derivatives 0 at $x = 0$.

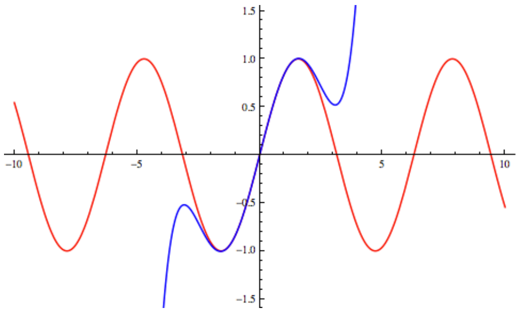
   Solution: $p(x) = x$.

   (b) This polynomial has $p(0) = 0,\ p'(0) = 1,\ p'''(x) = -1$, and all other derivatives 0 at $x = 0$.

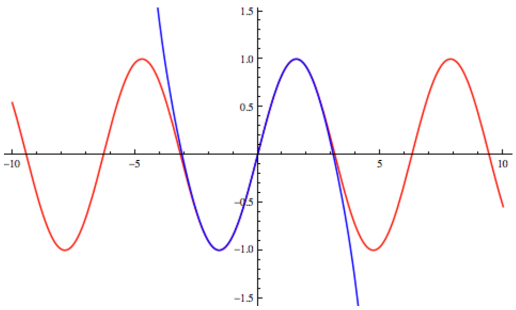
   Solution: $p(x) = x - x^3/6$.

   (c) This polynomial has $p(0) = 0,\ p'(0) = 1,\ p'''(x) = -1,\ p^{(5)}(x) = 1$, and all other derivatives 0 at $x = 0$.



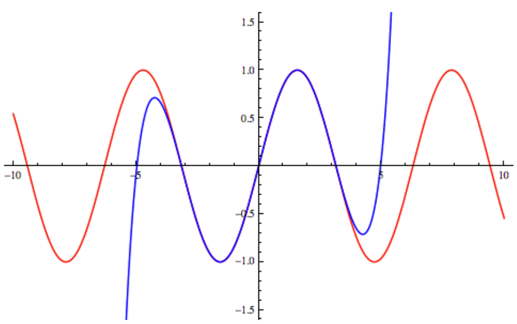
Solution: $p(x) = x - x^3/6 + x^5/120$.

(d) This polynomial has $p(0) = 0,\ p'(0) = 1,\ p'''(x) = -1,\ p^{(5)}(x) = 1,\ p^{(7)}(0) = -1$, and all other derivatives 0 at $x = 0$.



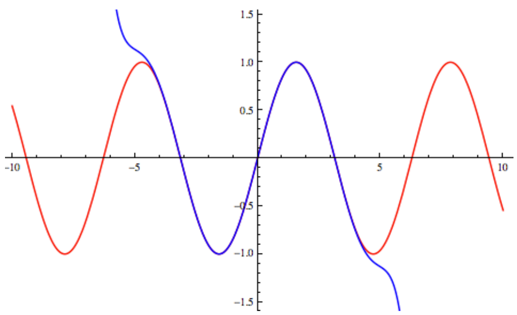
Solution: To avoid huge denominators, I will begin using factorial notation: $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7!$.

(e) This polynomial has $p(0) = 0,\ p'(0) = 1,\ p'''(x) = -1,\ p^{(5)}(x) = 1,\ p^{(7)}(0) = -1,\ p^{(9)}(0) = 1$, and all other derivatives 0 at $x = 0$.



Solution: $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7! + \frac{1}{9!}x^9$.

(f) This polynomial has $p(0) = 0,\ p'(0) = 1,\ p'''(x) = -1,\ p^{(5)}(x) = 1,\ p^{(7)}(0) = -1,\ p^{(9)}(0) = 1,\ p^{(11)}(0) = -1$, and all other derivatives 0 at $x = 0$.



Solution: $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7! + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11}$.

2. Let k be any positive integer. Find a function $p(x)$ such that $p^{(k)}(0) = 1$, but $p(0) = 0$ and all other derivatives of $p(x)$ are equal to 0 at $x = 0$.

   Solution: $p(x) = \frac{1}{k!}x^k$.

3. Let k be a any positive integer, and c be any real number. Find a function $p(x)$ such that $p^{(k)}(c) = 1$, but $p(c) = 0$ and all other derivatives of $p(x)$ are equal to 0 at $x = c$.

   Solution: $p(x) = \frac{1}{k!}(x - c)^k$.

4. Let $f(x)$ be any function. Write a formula (using either Σ notation of $\ldots$ notation) for a polynomial $P_n(x)$ which matches the value and first n derivatives of $f(x)$ at $x = 0$, but has all other derivatives equal to 0 at $x = 0$. This is the *degree n Taylor approximation of $f(x)$ centered at $x = 0$.*

   Solution:

$$
\begin{aligned}
P_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}}{n!}x^n \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k
\end{aligned}
$$

5. Let $f(x)$ be any function. Write a formula (using either Σ notation of $\ldots$ notation) for a polynomial $P_n(x)$ which matches the value and first n derivatives of $f(x)$ at $x = c$, but has all other derivatives equal to 0 at $x = c$. This is the *degree n Taylor approximation of $f(x)$ centered at $x = c$.*

$$\begin{aligned} P_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(c)}}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \end{aligned}$$

6. Approximate $\sqrt{5}$ by hand.

Solution: Naturally, there is no single correct answer, but here is what is intended. Consider the function $f(x) = \sqrt{x}$. We want to know $f(5)$. We happen to know $f(4) = 2$, and it is easy enough to compute derivatives of $f(x)$: $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$. Therefore we have:

$$\begin{aligned} f(4) &= 2 \\ f'(4) &= \frac{1}{4} \\ f''(4) &= \frac{1}{32} \end{aligned}$$

And therefore we have linear and 2^{nd} order Taylor approximations around $x = 4$ as follows.

$$\begin{aligned} P_1(x) &= 2 + \frac{1}{4}(x-4) \\ P_2(x) &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \end{aligned}$$

In particular, these give approximations of $\sqrt{5} = f(5)$.

$$\begin{aligned} P_1(5) &= 2 + \frac{1}{4} \\ &= 2.25 \\ P_2(5) &= 2 + \frac{1}{4} - \frac{1}{64} \\ &= 2 + \frac{15}{64} \\ &= 2.234375 \end{aligned}$$

The actual value of $\sqrt{5}$ is approximately $\sqrt{5} \approx 2.23606798$.