Taylor approximation

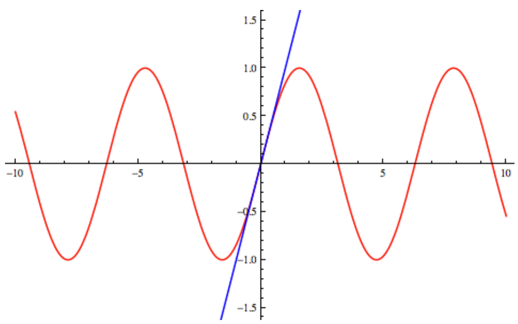
5 October 2011

Solutions can be found linked from:

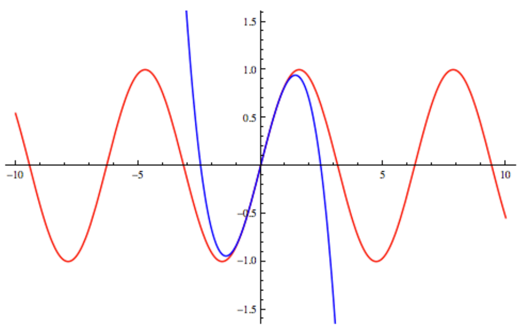
`http://math.harvard.edu/~pflueger/math1b.html`

1. The derivatives of the function $f(x) = \sin x$ at $x = 0$ follow the pattern: $1, 0, -1, 0, 1, 0, -1, \ldots$ (i.e. $f'(0) = 1$, $f''(0) = 0$, etc.). The following pictures show polynomials that have been rigged specifically to match these derivatives at 0. Determine which polynomial is shown in each picture.

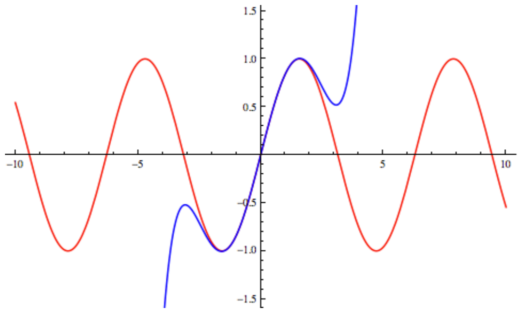
   (a) This polynomial has $p(0) = 0$, $p'(0) = 1$, and all other derivatives 0 at $x = 0$.



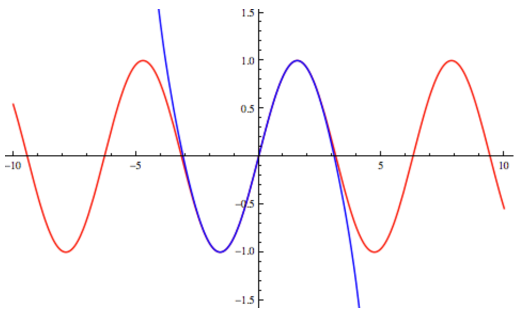
   (b) This polynomial has $p(0) = 0$, $p'(0) = 1$, $p'''(x) = -1$, and all other derivatives 0 at $x = 0$.



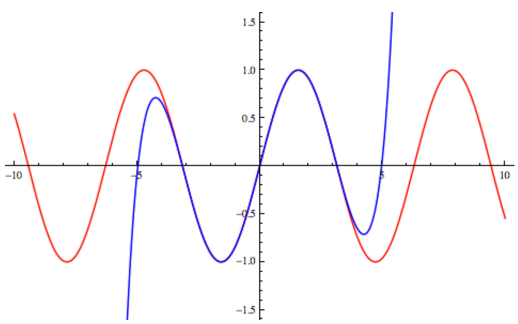
   (c) This polynomial has $p(0) = 0$, $p'(0) = 1$, $p'''(x) = -1$, $p^{(5)}(x) = 1$, and all other derivatives 0 at $x = 0$.



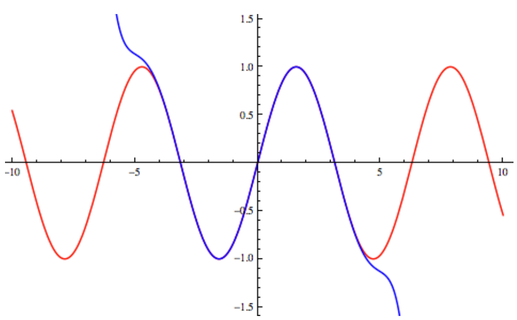
(d) This polynomial has $p(0) = 0$, $p'(0) = 1$, $p'''(x) = -1$, $p^{(5)}(x) = 1$, $p^{(7)}(0) = -1$, and all other derivatives 0 at $x = 0$.



(e) This polynomial has $p(0) = 0$, $p'(0) = 1$, $p'''(x) = -1$, $p^{(5)}(x) = 1$, $p^{(7)}(0) = -1$, $p^{(9)}(0) = 1$, and all other derivatives 0 at $x = 0$.



(f) This polynomial has $p(0) = 0$, $p'(0) = 1$, $p'''(x) = -1$, $p^{(5)}(x) = 1$, $p^{(7)}(0) = -1$, $p^{(9)}(0) = 1$, $p^{(11)}(0) = -1$, and all other derivatives 0 at $x = 0$.



2. Let k be any positive integer. Find a function $p(x)$ such that $p^{(k)}(0) = 1$, but $p(0) = 0$ and all other derivatives of $p(x)$ are equal to 0 at $x = 0$.

3. Let k be a any positive integer, and c be any real number. Find a function $p(x)$ such that $p^{(k)}(c) = 1$, but $p(c) = 0$ and all other derivatives of $p(x)$ are equal to 0 at $x = c$.

4. Let $f(x)$ be any function. Write a formula (using either Σ notation of $\ldots$ notation) for a polynomial $P_n(x)$ which matches the value and first n derivatives of $f(x)$ at $x = 0$, but has all other derivatives equal to 0 at $x = 0$. This is the *degree n Taylor approximation of $f(x)$ centered at $x = 0$.*

5. Let $f(x)$ be any function. Write a formula (using either Σ notation of $\ldots$ notation) for a polynomial $P_n(x)$ which matches the value and first n derivatives of $f(x)$ at $x = c$, but has all other derivatives equal to 0 at $x = c$. This is the *degree n Taylor approximation of $f(x)$ centered at $x = c$.*

6. Approximate $\sqrt{5}$ by hand.