Math 1B, lecture 6: Volumes of revolution

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1 Introduction

Solids of revolution provide a family of examples to study to understand slicing techniques, and the relationship between sums and integrals. Their main use, in my mind, is that they force you to be flexible, imagining several different integrals which compute the same physical quantity, and hence must be equal.

One prominent feature of volumes of revolution is that each can be computed in two rather different ways, depending on how the rotated region is sliced. Both ways slice the resulting solid. One produces washer-shaped slices, which would result from cutting the region into thin layers perpendicular to the axis of rotation. The other is quite different: it produces slices which are essentially the shapes that would result if very think layers were peeled off, as if by an apple peeler, by a blade touching the outside, as the solid rotates along its axis.

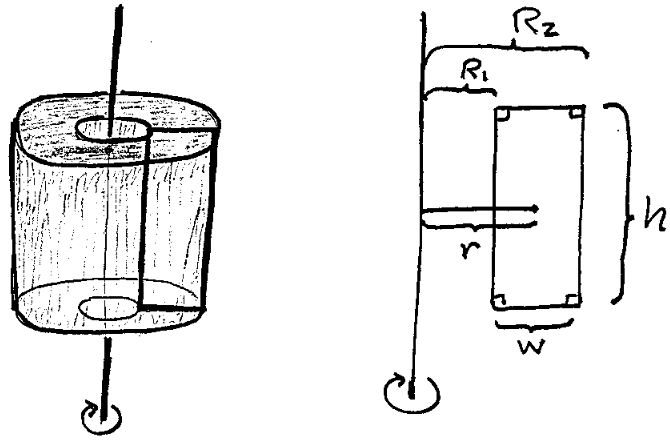
Solids of revolution also serve as an excellent example of a whole family of techniques that are generalized in multivariable calculus to broader situations. Essentially, one must eventually treat more complicated three-dimensional regions, and solids of revolution, with all of their built-in symmetry, are the best place to first cut your teeth. They also have the benefit of being able to be treated with only single-variable calculus.

The reading for today is Gottlieb, §28.1. The homework is problem set 4 (which includes weekly problems 1 and 2) and at topic outline. You should begin working on weekly problems $5, 6$ and 7.

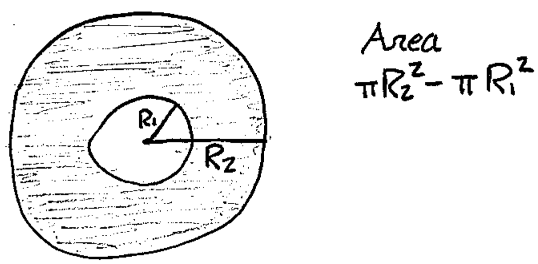
2 Revolving a rectangle

Both slicing strategies that we employ for solids of revolution essentially reduce to revolving rectangles about an axis parallel to one of their sides. There are two (equivalent) ways to express the area of the resulting solid; one is most useful when using shells, and one is most useful when using washers.

To fix notation: suppose that the rectangle is being revolved around an axis parallel to one of its pairs of sides. Let w, h be the dimensions of the rectangle itself. Suppose that the closer of these parallel sides is distance R_1 from the axis, and that the further is distance R_2 from the axis. Denote by r the average $\frac{1}{2}(R_1 + R_2)$; it is the distance from the axis to the center of the rectangle.



The volume of this solid of revolution can be computed as follows. It has height h, and its base is an annulus, shaped as follows.



This annulus has area $\pi R_2^2 - \pi R_1^2$, since this is the area of the large circle minus the area of the small circle.

Therefore, the area of the solid of revolution can be written as follows.

$$(\text{Volume}) = \pi(R_1^2 - R_2^2)h \tag{1}$$

Now, there is a second way to write this, that is more useful in some contexts. Recalling that r is the average of R_1 and R_2, i.e. $2r = R_1 + R_2$, and that the width of the rectangle is the difference of R_1 and R_2:

$$\begin{aligned} R_2^2 - R_1^2 &= (R_2 + R_1)(R_2 - R_1) \\ &= 2rw. \end{aligned}$$

Therefore, there is a second way to write the volume of this solid.

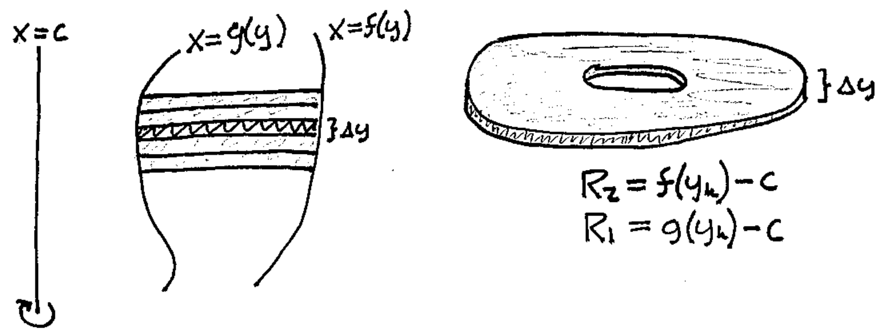
$$(\text{Volume}) = 2\pi rwh \tag{2}$$

Notice that this formula is rather intuitive: it simply says that the volume of this shape can be computed by computing the surface area of a cylinder with radius r and height h (this being simply $2\pi rh$) and multiplying by the thickness of the cylinder (in this case, w). This is fairly intuitive when w is very small (as it will be in the next section): you can imagine the solid as a rolled up sheet of paper; unrolling it gives a sheet of paper with length $2\pi r$, height h, and thickness w. It is less obvious that this formula should still hold when the solid is much ticker (and thus clearly cannot be unrolled). It is not clear, for example, what value to use for the radius r: the inner radius, the outer radius, or something in between? What we have just shown is that one can use the radius to the very center, and the correct volume is obtained.

Formulas 1 and 2 will be used, respectively, to compute volumes by washers and cylindrical shells. The difference depends on whether the revolved region is being sliced parallel to the axis of revolution or perpendicular to the axis of revolution.

3 Washers: perpendicular slices

Suppose that a region $\mathcal{R}$ is being revolving around an axis, and we wish to compute the volume of the resulting solid. The first way to do this is familiar from the previous class: slice the solid into pieces perpendicular to the rotation axis. This corresponds to slicing the region $\mathcal{R}$ into slices perpendicular to the axis of revolution.

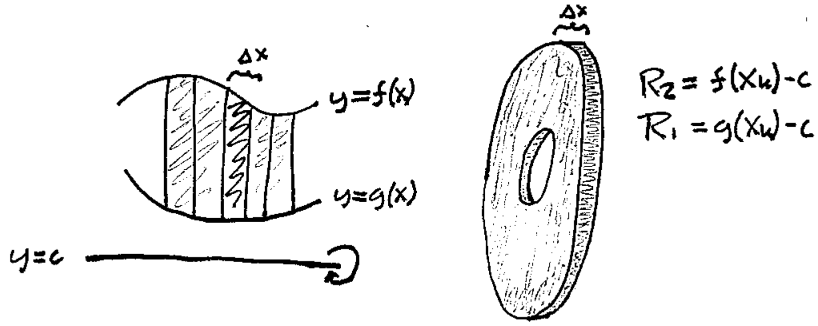


Thus each slice of $\mathcal{R}$ will produce a slice of the revolved solid which looks like a very flat annulus. It is necessary to sum the areas of all of these slices.

In order to obtain a formula, suppose that we are rotating about the line $x = c$, and the region is bounded on the side closer to the axis by $x = g(y)$, and on the side further from the axis by $x = f(y)$, for y ranging from a to b. Chop the region into n horizontal slices of width Δy, with endpoints $y_0, y_1, \ldots, y_n$ as usual. Then the k^{th} slice can be approximated as a rectangle with dimensions $f(y_k) - g(y_k)$ and Δy. From the discussion in the previous section, the result of rotating such a rectangle can be written as $\pi(R_2^2 - R_1^2)\Delta y$. Now, the radii R_1, R_2 are given by the distances from the edges of this very thin rectangle to the axis, which should just be $g(y_k) - c$ and $f(y_k) - x$, respectively. Therefore the volume of the solid can be approximated by this sum, which can then be written as an integral.

$$
\begin{aligned}
\text{(Volume)} &\approx \sum_{k=1}^{n} \pi \left[(f(y_k) - x)^2 - (g(y_k) - c)^2\right] \Delta y \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \pi \left[(f(y_k) - x)^2 - (g(y_k) - c)^2\right] \Delta y \\
&= \int_a^b \pi \left[(f(y) - c)^2 - (g(y) - c)^2\right] dy
\end{aligned}
$$

Of course, the exact same discussion could be carried out if the axis of rotation were horizontal, and the region were bounded by functions $f(x)$ and $g(x)$ of x.

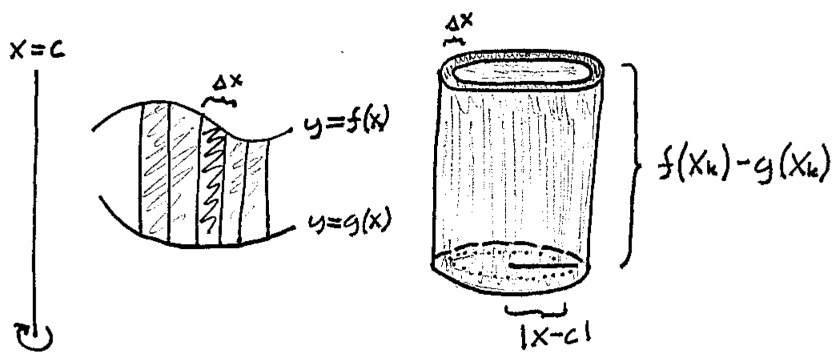


$$\text{(Volume)} = \int_a^b \pi\left[(f(x)-c)^2-(g(x)-c)^2\right]dx$$

4 Shells: parallel slices

Suppose now that the region $\mathcal{R}$ is sliced parallel to the axis of revolution. Now each slice, upon revolution, becomes a very thin cylinder.

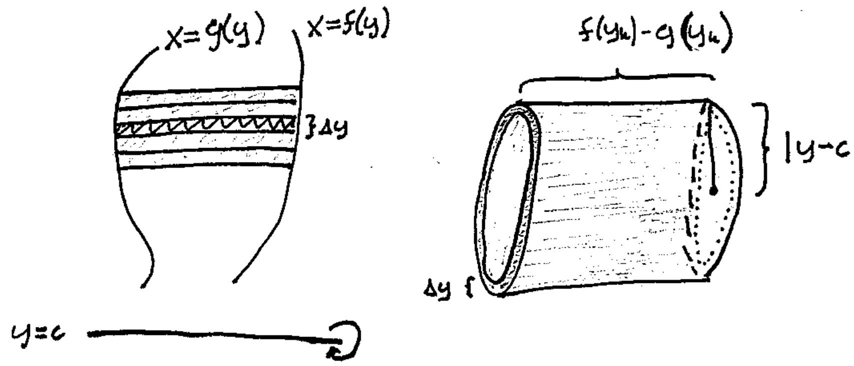
To obtain a formula, suppose that we are rotating about the line $x=c$, and the region is bounded below by $y=g(x)$, and above by $y=f(x)$, where x ranges from a to b. Chop the regio into n vertical slices as usual. This time, the result of rotating the k^{th} slice is a very thin cylinder.



Now we should use formula 2 from the previous section. The radius r is equal to the difference between c (the x value of the axis) and $\frac{1}{2}(x_{k-1}+x_k)$ (the x value of the middle of the slice). Thus the volume of the rotation of this slice can be approximated by $2\pi\left|\frac{x_{k-1}+x_k}{2}-c\right|(f(x_k)-g(x_k))\Delta x$. For a large number of slices, the difference between x_k and x_{k-1} is very small, and so they are basically interchangeable, and both are interchangeable with their average in an approximation for the k^{th} slice (this was the reason why left, right, and midpoint approximation all converge to the same integral; it is true, however, that some may converge faster than others). Therefore we can simplify matters by simply approximating the volume of the revolution of the k^{th} slice by $2\pi\left|x_k-c\right|(f(x_k)-g(x_k))\Delta r$. Summing and taking the limit:

$$
\begin{aligned}
(\text{Volume}) &\approx \sum_{k=1}^{n} 2\pi \left| x_k - c \right| (f(x_k) - g(x_k))\Delta r \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi \left| x_k - c \right| (f(x_k) - g(x_k))\Delta r \\
&= \int_a^b 2\pi \left| x - c \right| (f(x) - g(x))dx
\end{aligned}
$$

A similar formula is obtained for revolving a region about a horizontal axis $y = c$ and chopping the region horizontally. Then again, each slice revolves to produce a thin cylinder.



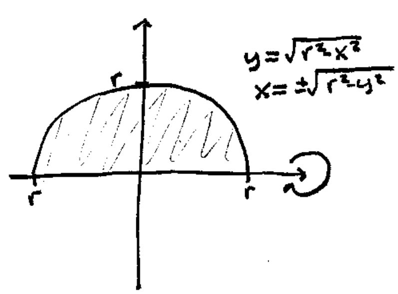
The integral in this case is:

$$(\text{Volume}) = \int_a^b 2\pi \left| y - c \right| (f(y) - g(y))dy$$

5 Examples

Example 5.1. A sphere of radius r.

A sphere of radius r can be obtained by revolving the region between $y = \sqrt{r^2 - x^2}$ and the x axis around the x axis. This can be done in two ways: slicing horizontally or vertically.



First, consider vertical slicing of the semicircle. Here, x ranges from $-r$ to r, so for n slices we can take $x_0 = -r, \Delta x = 2r/n, x_k = x_0 + k\Delta x$. The k^{th} rectangle can be approximated as having width Δx and height $\sqrt{r^2 - x_k^2}$. The bottom edge of this rectangle touches the axis of revolution, so upon rotation it produces a disc of volume $\pi(\sqrt{r^2 - x_k^2})^2 \Delta x = \pi(r^2 - x_k^2)\Delta x$. Summing and taking the limit, the volume of the sphere is:

$$\begin{aligned}
\text{(Volume)} &\approx \sum_{k=1}^{n} \pi(r^2 - x_k^2)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \pi(r^2 - x_k^2)\Delta x \\
&= \int_{-r}^{r} \pi(r^2 - x^2)dx \\
&= \pi\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

Next, consider horizontal slicing of the semicircle, which will produce cylindrical shells. Here, y ranges from 0 to r, and the cross section at y consists of x values ranging from $-\sqrt{r^2 - y^2}$ to $\sqrt{r^2 - y^2}$. Thus the k^{th} slice can be approximated by a rectangle of height Δy, width $2\sqrt{r^2 - y_k^2}$, and distance y_k from the axis. Upon revolution, this produces a cylinder of radius y_k, length $2\sqrt{r^2 - y_k^2}$, and thickness Δy. Such a cylinder has volume $2\pi y_k \cdot 2\sqrt{r^2 - y_k^2} = 4\pi y_k \sqrt{r^2 - y_k^2}$. Summing and taking the limit:

$$\begin{aligned}
\text{(Volume)} &\approx \sum_{k=1}^{n} 4\pi y_k \sqrt{r^2 - y_k^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} 4\pi y_k \sqrt{r^2 - y_k^2} \\
&= \int_{0}^{r} 4\pi y \sqrt{r^2 - y^2} dy
\end{aligned}$$

Evaluating this by substitution gives the same volume, $\frac{4}{3}\pi r^3$.

I may add more examples to this list, however a good list of examples with solutions can be found on Janet's worksheet for today's class, which is located on the course website under "Additional Resources," in the subcategory "worksheets," and listed as the "volumes of revolution" worksheet in the first column. These are the examples which we used in class.

6 Appendix: The theorem of Pappus

I mention here a theorem often attributed to the fourth century Greek mathematician Pappus of Alexandria, sometimes called "Pappus's centroid theorem." It gives a very fast an intuitive way to obtain the volume of a solid of revolution, if the region being revolved is reasonably symmetric.

Theorem 6.1. *Suppose that a region $\mathcal{R}$ is revolved around an axis to form a solid. Then the volume of this solid is*

$$\textit{(Volume)} = 2\pi r \cdot A$$

where r is the distance from the axis to the center of mass of $\mathcal{R}$, and A is the area of R.

The proof is a direct computation, using cylindrical shells. The trickiest part of it, in fact, is understanding how to compute the center of mass of a region, which I shall not get wrapped up in right now.

Another way that this is sometimes stated is that the volume is equal to the area of $\mathcal{R}$ times the distance travelled by $\mathcal{R}$ in the rotation.

The reason that this theorem is usually only useful when the region $\mathcal{R}$ is nearly symmetric is that otherwise is is probably just as difficult to compute the center of mass of the region as it would be to compute the volume of revolution. However, this does give a very fast way to find the volume of a torus, for example.

Corollary 6.2. *The volume of a torus formed by rotating a circle of radius* r *around an axis that is distance* R *from the axis is equal to* $2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R$.