Lecture 34: Phase plane analysis II

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1 Introduction

This lecture is a continuation of the previous lecture. The object of study is still systems of differential equations which describe pairs of functions, which will be studied in the context of the phase plane. Today we cover a technique which can sometimes be used to solve systems of equations, which is analogous to the technique of separation of variables. This technique relies on the following idea: if $x(t), y(t)$ are two functions of t, then sometimes we can view y and a function of x in order to understand what the trajectory will look like in the $x-y$ plane. This technique sometimes will give explicit formulas, but it will more often give what is called an implicit formula.

There is no new reading for today (this material is contained in Gottlieb §31.5, which was assigned last time), although I recommend reading the solutions to problem 6 on the worksheet "Systems and shapes of trajectories." The homework is problem set 33 and a topic outline, plus the remaining weekly problems: $29, 30, 31$, and 32. Note that the course schedule also indicates that problem set 34 is due, but this problem set cossets only of the last three weekly problems.

2 Solving with the chain rule

The technique of using the chain rule to solve for solution trajectories is well-illustrated by the Romeo and Juliet system that we have discussed in the previous two classes.

$$\begin{cases} r' = j \\ j' = -r \end{cases}$$

Now, the basic idea is to think of r as a function not of time t, but of j. Now note that by the chain rule, $r'(t) = \frac{dr}{dj} j'(t)$, and therefore $\frac{dr}{dj} = \frac{r'(t)}{j'(t)}$. Now, given the system we are solving, the right side of this can be written as $-\frac{j}{r}$. So we have the following differential equation for r as a function of j.

$$\frac{dr}{dj} = -\frac{j}{r}$$

Now this can be attacked using separation of variables.

$$\begin{aligned} r dr &= -j dj \\ \int r dr &= -\int j dj \\ \frac{1}{2} r^2 &= -\frac{1}{2} j^2 + C \\ r^2 + j^2 &= 2C \end{aligned}$$

Therefore it follows from this that, for any solution trajectory, the distance $\sqrt{r(t)^2 + j(t)^2}$ to the center of the phase plane must be constant at all times. So this give another way to see that the solution trajectories are in fact circles around the origin in phase space.

The general technique looks like this. Begin with a system of differential equations of the following form.

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

Now, this could also be written using Leibniz notation.

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

Now viewing y as a function of x, the chain rule states that $\frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt}$. In other words, $\frac{dy}{dx}f(x, y) = g(x, y)$, and therefore whenever $f(x, y) \neq 0$, $\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$. Now, if this differential equation can be solved (for example, by separation of variables), then it may be possible to understand the shape of the solution trajectories.

3 Examples

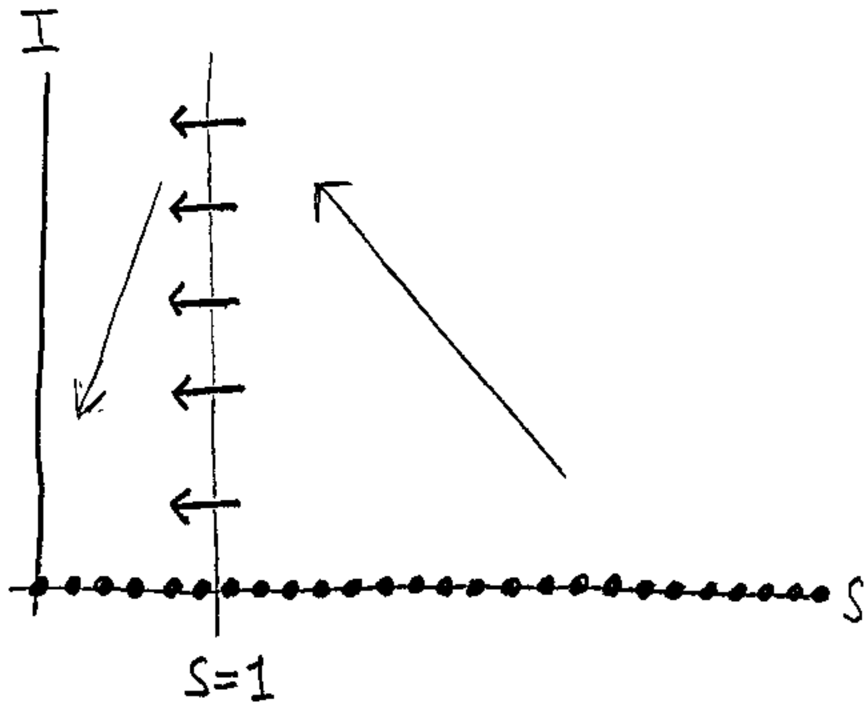
I will probably add a couple more examples to this document later today. For now, here is one example.

Consider the following model for the spread of a fatal epidemic. Here $S(t)$ measures the number of susceptible people (presumably countered in thousands some larger unit size), and $I(t)$ measures the number of infected people.

$$\begin{cases} S' = -IS \\ I' = IS - I \end{cases}$$

The idea behind this model is that new infections occur at a rate proportional the number of times an infected person meets a susceptible person, i.e. proportional to I times S. Each infection decreases S and increases I by the same amount. In addition, some number of infected people die each day, and this number is proportional to the number of infected people.

A qualitative analysis of this system is shown below, using the fact that $S' = 0$ whenever $I = 0$ or $S = 0$, and $I' = 0$ whenever $I = 0$ or $S = 1$.

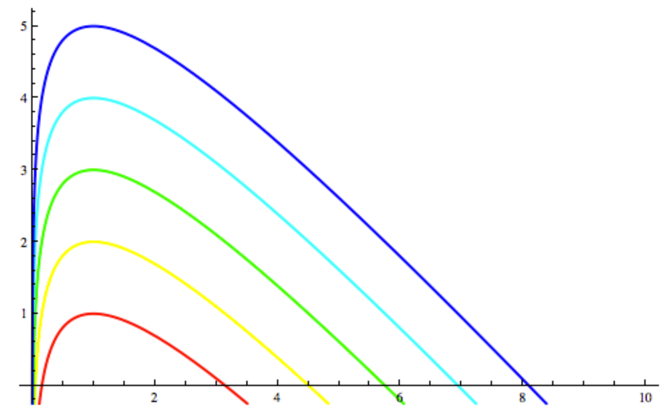


Notice that unlike other examples that I have considered in my notes so far, there are infinitely many equilibrium points in this system, namely all places where $I = 0$.

The chain rule technique can be applied to this problem, as follows.

$$\begin{aligned}\frac{dI}{dS} &= \frac{I'}{S'} \\ &= -1 + \frac{1}{S} \\ dI &= (-1 + \frac{1}{S})dS \\ I &= -S + \ln S + C\end{aligned}$$

Therefore, the solution curves will look precisely like the graph of the function $f(x) = -x + \ln x$, translated vertically by horizontal contants. Several such trajectories are shown below.



Note that although this graph shows these trajectories passing though the horizontal axis, in fact they should end at the axis itself, where equilibrium takes place.

Observe what this reveals, qualitatively speaking: the course of the epidemic will always follow roughly the same arc: a very large number of people will become infected, leaving very few left uninfected, and then these people will die off almost entirely as a few more of the survivors also become infected. The result is a vastly decreased, but not quite zero, population.