Lecture 33: Phase plane analysis I

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1 Introduction

This lecture and the next focus on the analysis of systems of differential equations of the form considered last time. The main setting in which we will study these systems is by analyzing the *phase plane*, which is a plane whose points represent possible pairs of values for the two functions being studied. These techniques will be almost entirely qualitative, concerning the general shape and long-term behavior of solutions, rather than their precise values.

The basic strategy in analyzing the phase plane is to first identify the points where solution trajectories must be exactly horizontal or exactly vertical. This will partition the phase plane into a number of regions, and the general direction of trajectories in each region can be identified. From here it is often possible to see the general behavior of trajectories, even without explicitly solving for them.

These notes will have some images later, once I get a change to draw some and scan them.

The reading for today is Gottlieb §31.5. The homework is problem set 32 and a topic outline. You should begin working on weekly problems 30 and 31.

2 The phase plane and trajectories

We are studying systems of differential equations of the following form. Here, x, y both denote functions of a variable t (so really they are shorthand for $x(t), y(t)$)

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

One way to understand a system like this is by trying to determine what the trajectories of the solutions look like in the phase plane. Recall that the phase plane is a plane with coordinates x and y, and a trajectory is obtained by plotting all the points $(x(t), y(t))$ of some solution to the system.

There are two critical facts about trajectories in the phase plane.

- Through any point in the phase plane, there is a unique solution trajectory.
- No two solution trajectories can intersect.

Also recall that there are *equilibrium solutions*, whose entire trajectories consist of a single point. These points can be identified as the values of x, y such that $f(x, y) = 0$ and $g(x, y) = 0$, so that the trajectory does not move at all.

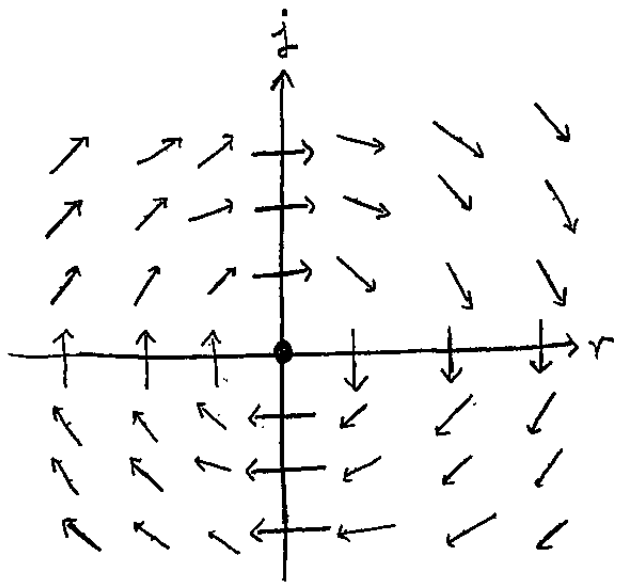
For example, recall the toy example of Romeo and Juliet from last time.

$$\begin{cases} r'(t) = j(t) \\ j'(t) = -r(t) \end{cases}$$

Then the trajectories in the phase plane consist of counter-clockwise circles, plus the single point at the origin (which is an equilibrium solution).

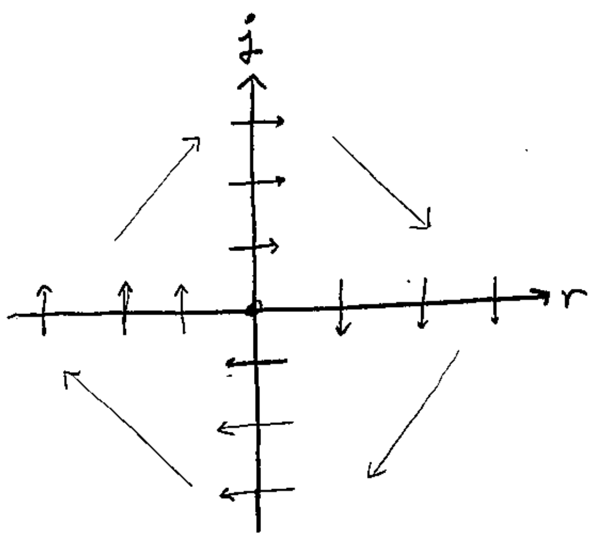
If we do not yet know what the trajectories look like, one technique, which is entirely analogous to drawing slope fields[1], is to observe that at any point (x, y), the values $f(x, y), g(x, y)$ identify the direction in which the solution trajectory through (x, y) must be traveling. Indeed, the trajectory will have slope $g(x, y)/f(x, y)$ (rise over run), unless $f(x, y) = 0$ and $g(x, y) \neq 0$, in which case the trajectory will be perfectly vertical.

For example, here is what happens if we draw arrows in various places in the phase plane for the Romeo and Juliet example, indicating which way trajectories must be traveling.



From this picture (especially if it were drawn more carefully), you could conjecture that the trajectories will move in counterclockwise circles.

Another, more readable way to encode some of this information is to simple draw the following picture.



[1]Indeed, slope fields are a special case of this technique, wherein $x(t) = t$.

Here, I have indicated the places where the trajectories must be perfectly horizontal (and whether they are moving left or right) and the places where they must be perfectly vertical. This partitions the plane into four regions. In each of these regions, I have drawn a single arrow to indicate which of the four general directions the trajectories must be traveling (up-right, up-left, down-left, or down-right). I can determine this information just by looking at the signs of $f(x, y)$ and $g(x, y)$ in these regions.

While this second picture does convey less information (I know nothing about how steep the trajectories are), it turns out to still contain some useful information. More importantly, it is much easier to draw these sorts of figures that the full direction field picture in more complicated situations. The following sections describe the general process for doing this.

3 A broad outline of phase plane analysis

The following steps are the basic steps of so-called phase plane analysis. Certainly these will not be done in exactly this order in all situations, but this is a good general approach. These steps will be described in more detail in subsequent sections.

1. Identify the places where $x' = 0$. These are called null clines. Draw vertical hashes at these places.
2. Identify the places where $y' = 0$. These are also called null clines. Draw horizontal hashes at these places.
3. Mark the places where the null clines of both types meet as equilibrium points.
4. Away from the equilibrium points, determine whether the trajectory must be moving left, right, up or down by checking the sign of the other function's derivative.
5. Orient each region that remains between the null clines by determining the sign of x' and y' in these regions.

4 Null clines

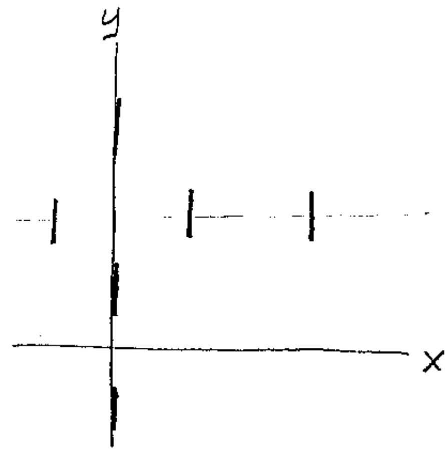
The first step in phase plane analysis is to find the so-called null clines[2].

For example, consider the following system, which could model a simple predator-prey interaction.

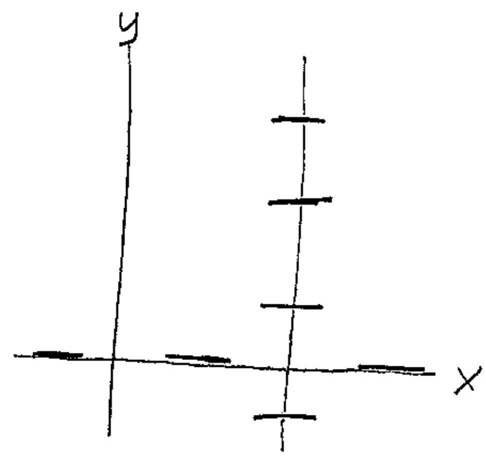
$$\begin{cases} x' = x - xy \\ y' = -y + xy \end{cases}$$

Then $x' = x(1 - y)$, hence $x' = 0$ if and only if either $x = 0$ or $y = 1$. Hence, the x-null cline consists of all points where $x = 0$ or $y = 1$. We can put vertical hashes in all of these places, since $x' = 0$ in such places, hence the trajectory must be vertical.

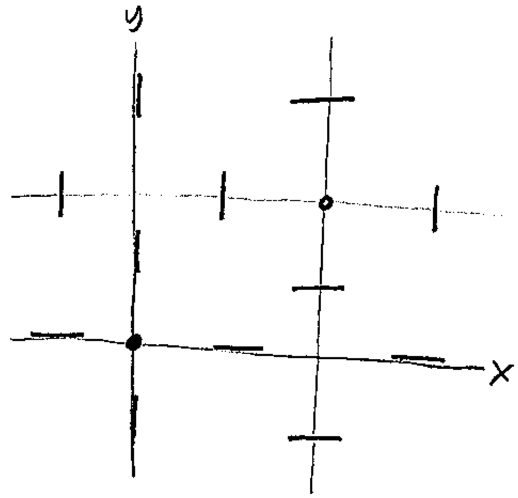
[2]The word "cline" is apparently a combination of the words "circle" and "line." Presumable it is used because a cline is a sort of curve that might be a circle, line or something else. Why these are called clines, rather than curves or anything else, is a mystery to me, but it seems to be conventional in this field.



Next, consider the places where $y' = 0$. Now $y' = -y(1-x)$, so this occurs wherever $y = 0$ or $x = 1$. So these lines comprise the y-null cline, on which we can draw horizontal hashes.



Now, if all of these are draw on the same figure, we see that the two null clines intersect in precisely two places: $(0,0)$ and $(1,1)$. So these are the places where the trajectory is not moving at all; they can be marked as equilibria by dots, and the other hashes left in place.

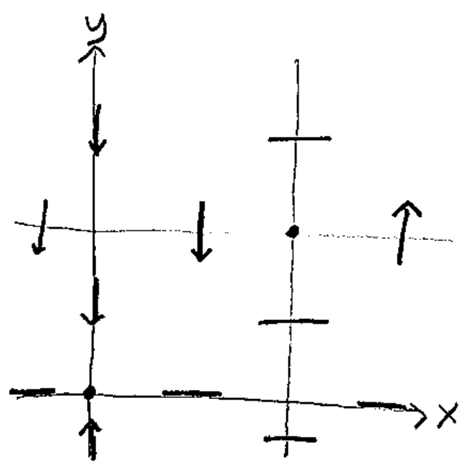


5 Orienting null clines

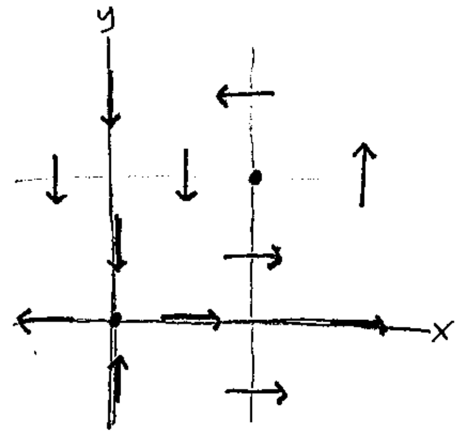
The next step is to "orient" the null clines. Here the objective is to determine, for the horizontal and vertical hashes drawn, whether the trajectory will be moving up, down, left, or right. This is done, naturally, by examining the sign of y' (for the x-null cline) and the sign of x' (for the y-null cline).

The critical fact is that the sign of x' will change only upon crossing the x-nullcline. It does not always need to change when crossing the null cline, but it certainly will not change otherwise.

In the predator-prey example, we can begin by orienting the vertical hashes along the y-axis: here, $y' = -y + xy = -y$, so the trajectories must be moving downwards for $y > 0$, and upwards for $y < 0$ (note that the places where $y > 0$ are the only places relevant to the actual predator-prey model, but I include the other three quadrants just to illustrate what this model would suggest if negative values were permitted).



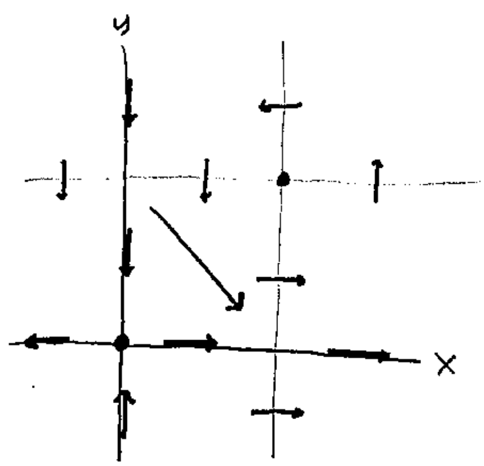
Now, similarly, all the other vertical hashes on the same side of the y-null cline must also point downward. As for the rest of the vertical hashes, notice that upon crossing $x = 1$, the value $y' = -y(1 - x)$ will change from negative to positive, hence the hashes on the other side must be pointing up. Applying similar reasoning, we can orient all of the horizontal hashes in a similar manner.



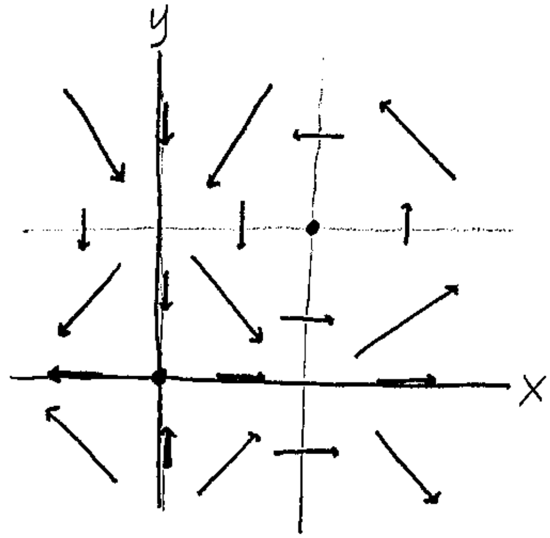
6 Orienting regions

The next step, after orienting all of the arrows along the null clines in the phase plane, is to orient all the remaining regions. This amounts to determining the sign of x' and the sign of y' in those regions. Now, either of these signs can change only when crossing the relevant null cline. Hence it suffices to test a single value from each region to learn the orientation.

For example, at $(x, y) = (0.5, 0.5)$, $x' = 0.25$ and $y' = -0.25$, so the trajectory is traveling down-right. Since signs of derivatives can change only on crossing null clines, this must be the orientation for the entire region around $(0.5, 0.5)$. This can be indicated with a single arrow as shown.



There are several regions remaining. By choosing any particular point in each region and determining what x' and y' must be at that point, these regions can be oriented as well. The result is shown.



The resulting figure now shows most of the easily-accessible qualitative information about this system of differential equations. For example, this phase plane analysis shows that solutions to this predator-prey system have trajectories that orbit around the equilibrium $(1, 1)$. Unfortunately, the phase plane analysis cannot identify all information: it remains unknown (from this method) whether the solution will create closed orbits (as at the left), spirals (as at the right) or something more exotic yet. Different methods would be needed to understand this difference (one will be seen in the next class).

Nevertheless, some important qualitative information does come through. There is a stable point at $(1, 1)$, and when the two populations are not at this point, they will go through cycles of the following sort:

1. There are too many predators, causing the prey population to shrink (due to predation), as well as the predator population (due to lack of food).

2. The predator population will shrink enough that the prey population will begin to grow again; the predators will continue to shrink since the food source is still scarce.

3. The prey will become plentiful enough that the predators begin to grow in population. The prey's growth will slow, but continue to grow for some time.

4. The predators will become plentiful enough the population growth of the prey stops, and the prey population begins to contract.

5. The predator population will continue to grow to excessive levels until the prey population shrinks too far, and the cycle begins again.

Other models (symbiosis or competition, with or without limited resources) would lead to a number of other sorts of patterns; sometimes there will be cyclic behavior like this, or sometimes one or both species will be pushed to extinction. Some cases of this all be explored next class and on the homework.