# Lecture 32: Systems of equations

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## 1 Introduction

So far in the differential equations unit, we have considered individual differential equations which describe how a single function changes over time. The last three lectures will give a brief introduction to systems of differential equations. The difference will be that we will now study several interacting quantities changing over time. The interaction is modeled by a system of differential equations.

The main examples for these three classes will be population dynamics. We have discussed how to model a single population in various ways, but suppose instead that you wish to model two populations, one of which feeds on the other? This leads to so-called predator-prey models.

Just as in the unit so far, there will be three aspects to the study of systems: modeling (i.e. choosing a system that describes a particular situation), qualitative analysis (determining equilibria, long-term behavior, and general "shape" of solutions), and explicit solution. We will focus primarily on qualitative analysis when studying systems. This is mainly due to the difficulty of solving systems explicitly. The one situation in which situations can be studied rather explicitly is the case of linear homogeneous systems, but these require knowledge of linear algebra to study effectively; these are considered in Math 21B.

The reading for today is the handout "introduction to systems of differential equations." The homework is problem set 31 and a topic outline. You should begin working on weekly problem 29.

## 2 A simple system

The following example is attributed to Steven Strogatz, a former teaching fellow in Math 1B and now a professor of applied mathematics at Cornell[1]. It illustrates some of the basic aspects of systems in a very simple context. I shall discuss this example in some detail, attempting to highlight the main aspects of the study of systems of differential equations.

Two Italian teenagers, let us call them Romeo and Juliet, have a rather dysfunctional relationship. We can model their feelings for each other using two functions. Here, the variable $t$ measures the number of days from some particular starting point.

$$\begin{aligned} r(t) &= \text{Romeo's interest in Juliet at time } t \\ j(t) &= \text{Juliet's interest in Romeo at time } t \end{aligned}$$

This definition illustrates the starting point of systems of differential equations: we are modeling a situation that evolves over time using not just one function, but two. Naturally, these two functions will be intertwined with each other.

Naturally, these two functions might not be constant: times change, interest grows and wanes. In this case, let us imagine (not improbably) that Romeo will begin to care for Juliet more the more she is interested

[1] Strogatz recently wrote a series of columns for the New York Times on mathematics for a broad audience, which you may enjoy looking up on Google

in him. In other words, the more Juliet likes him, the more rapidly his interest grows. This could be expressed in the following equation.

$$r'(t) = j(t)$$

Juliet, on the other hand, has the unenviable tendency to lose interest in the people that are interested in her. In other words, the more Romeo likes her, the more quickly she loses interest, while her interest grows in response to Romeo's disinterest. This could be expressed in the following equation.

$$j'(t) = -r(t)$$

The pair of these two equations is a *system of differential equations*.

$$\begin{cases} r'(t) = j(t) \\ j'(t) = -r(t) \end{cases}$$

What we are interested in modeling is the *pair* of both $r(t)$ and $j(t)$. The system expresses, in the conjunction of two equations, how this pair moves at any given moment.

In order to "solve" a system of differential equations, it is necessary to determine both of these functions simultaneously. In this case, we can find all of the solutions fairly easily with an observation. Since $r'(t) = j(t)$, it is also true (by differentiating) that $r''(t) = j'(t)$. But from the second of the two equations in the system, $j'(t) = -r(t)$. Therefore $r''(t) = -r(t)$. Now this is a familiar second-order differential equation. Its general solutions can be written a follows.

$$r(t) = C_1 \sin t + C_2 \cos t$$

It turns out that finding this expression for $r(t)$ is also good enough to find $j(t)$. In fact, $j(t) = r'(t)$. Therefore, once we have chosen any two constants $C_1, C_2$, we can take $r(t)$ to be the expression above and deduce the following.

$$j(t) = C_1 \cos t - C_2 \sin t$$

Notice that indeed if $j(t)$ is given in this way, then $j'(t) = -C_1 \sin t - C_2 \cos t = -r(t)$. So $r'(t) = j(t)$ and $j'(t) = -r(t)$, so for any constants $C_1, C_2$, these two expressions give a solution to the system.

In fact, we have found in this way what is called the *general solution* of the differential equation.

$$\begin{cases} r(t) = C_1 \sin t + C_2 \cos t \\ j(t) = C_1 \cos t - C_2 \sin t \end{cases}$$

Notice that the general solution has two arbitrary constants $C_1, C_2$, and that these two constants determine two different functions. These two functions, taken together, solve the system of differential equations.

*Observation* 2.1. This observation is not needed for this course, but may interest some of you. If we are willing to work in the slightly more mysterious world of complex numbers, this system and its solutions can be written in an intriguing way. Both functions $r(t), j(t)$ could be jointly expressed by a single complex function $s(t) = r(t) + ij(t)$, where $i$ is a square root of $-1$. Then the system $r'(t) = j(t), j'(t) = -r(t)$ can be expressed in a single equation: $s'(t) = -is(t)$. Therefore, this can be solved in the usual way, giving $s(t) = Ce^{-it}$. Here, $C$ is a *complex* constant; you could write it $C = C_1 + iC_2$, where $C_1$ and $C_2$ are real. Then $s(t) = (C_1 + iC_2)(\cos t - i \sin t) = (C_1 \cos t + C_2 \sin t) + i(-C_1 \sin t + C_2 \cos t)$. So the complex general solution in fact is the same as the general solution to the system found above. This observation shows one way that complex numbers frequently come up in practice: a complex number records two real numbers that are harnessed to each other in some meaningful way, and economically encodes this relationship.

Now, to make this more concrete, lets consider a particular solution. As usual, this will be determined by an appropriate initial condition. The initial condition must consist of the initial state of the system, i.e. the initial interest of Romeo in Juliet and Juliet in Romeo. Let's suppose that initially Romeo likes Juliet, and Juliet is indifferent.

$$\begin{cases} r(0) = 1 \\ j(0) = 0 \end{cases}$$

In order to find the solution to the system satisfying this initial condition, we must determine appropriate values of $C_1, C_2$. That is, we must choose $C_1, C_2$ so that:

$$\begin{aligned} C_1 \sin 0 + C_2 \cos 0 &= 1 \\ C_1 \cos 0 - C_2 \sin 0 &= 0 \end{aligned}$$

This is not difficult: we find that $C_2 = 1$ and $C_1 = 0$. Therefore we have found the particular solution to this initial value problem.

| Initial value problem | Solution |
|---|---|
| $\begin{cases} r'(t) = j(t) \\ j'(t) = -r(t) \end{cases}$ | $r(t) = \cos t$ |
| $\begin{cases} r(0) = 1 \\ j(0) = 0 \end{cases}$ | $j(t) = -\sin t$ |

So in fact, if Romeo and Juliet have interest in each other governed by this model, they are doomed to continue cycling between love and hate forever.

Section 5 discusses how to visualize this solution in two different ways.

# 3 The basic notions of systems

A system of differential equations is, in general, a collection of differential equations involving several different functions. For this class, we will only consider first order systems (i.e., only first derivatives in the equations) involving two functions. The systems that we study will have he following form. Here, $x$ and $y$ are both regarded as begin functions of a variable $t$, and $f, g$ are some functions of two variables.

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{cases}$$

As long as it is understood that $x$ and $y$ are both functions of a variable $t$, this can be written more readably as follows.

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

The point is that $x$ and $y$ are two different quantities (such as people's interest in each other, or sizes of certain populations), each of which is changing with time in a way that is determined by the quantities $x$ and $y$.

An *initial value problem* for a system of differential equations like this consists of the system of differential equations together with an initial state. The initial state, in this case, simply consist of the data of what these two quantities are at time 0. Here $x_0, y_0$ are some constants.

$$\begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

An initial value problem has a unique solution, meaning a pair of functions $x(t), y(t)$ which solves it.

The *general solution* of a system of differential equations consists of all *pairs* of functions $x(t), y(t)$ that solve the system. In other words, the general solution consists of a family of solutions that contains a solution to every possibly initial condition for the system.

One final bit of terminology: a solution to a system will be called an *equilibrium solution* if it consists of two constant functions. Just as with individual differential equations, an equilibrium solution consists of values of the modelled quantities that are in balance, so that nothing will change over time.

# 4 Interacting population modeling

A more interesting example of a system of differential equations comes from modeling a pair of animal populations which interact in some way. There are several reasons this might happen. The species could be in *competition* (there are limited resources, so the more of the other species are present, the harder it is to survive), *symbiosis* (each population helps the other, for example by scaring off predators or cooperating to obtain food), or a *predator-prey relationship* (one population helps the other survive, to its own detriment).

As a first example, suppose that there are two populations: worms and birds. Let their respective sizes at time $t$ be $w(t)$ and $b(t)$.

First focus on the worm population, assuming that there are no birds. Perhaps they have plenty of food, and so their population grows exponentially: $w'(t) = 0.01w(t)$. But now suppose that there are some birds, which begin eating the worms. Naturally, this will decrease the worm population, in a way that is proportional to the number of birds around. This could be modeled in a rather straightforward way by $w' = 0.01w - 0.001bw$.

What about the bird population? With no worms present, we could imagine that the birds will die off exponentially due to lack of food: $b' = -0.02b$. But on the other hand, the more worms are present, the better the birds will do, and the more quickly will they grow. So we could also model this by the addition of a $bw$ term: $b' = -0.02b + 0.005bw$. This gives the following system of differential equations.

$$\begin{cases} w' = 0.01w - 0.001bw \\ b' = -0.02b + 0.005bw \end{cases}$$

This sort of system is a *predator-prey model*, and is a good first approximation of the behavior of two species with this sort of interaction.

Notice that this system has some equilibria. Recall that an equilibrium solution is a constant solution; in this was it amounts to values for $w$ an $b$ which ensure that $w' = 0$ and $b' = 0$. Now, $w' = 0$ if and only if either $w = 0$ (there are no worms) or $0.01 - 0.001b = 0$, i.e. $b = 10$. So if there are ten birds, then the birds will eat the worms exactly as quickly as they can reproduce. On the other hand, $b' = 0$ if and only if $b = 0$ or $-0.02 + 0.005w = 0$, i.e. $b = 0$ or $w = 4$. Therefore, there are exactly two equilibria for the system as a whole: either there are 4 worms and 10 birds, or no birds or worms at all. In both cases, both populations exactly sustain themselves.

Note that there are many variations on this model, taking different potential features of the situation into account. Here are some examples, although you an surely imagine many more.

- The birds could have a food source besides worms. To reflect this, the constant $-0.02$ could be replaced be a positive constant, since the birds will grow even when there are no worms.
- The two species might be two symbiotic species, instead of predator and prey. Then both equations in the system would have $wb$ terms with positive coefficients, since each population causes the other to grow.
- The two species could be competing: then both equations would have a negative $wb$ term.
- The worm population in the absence of birds might be modeled logistically, to reflect limited food sources creating a carrying capacity. For example, the $0.01w$ term in the first equation could be replaced by $0.01w(1 - 0.0001w)$, while leaving the same "competition term" in place.

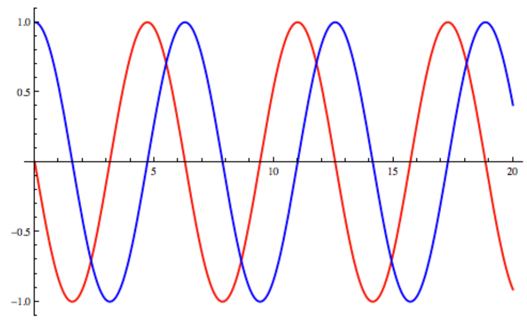
There are many different combinations of these sorts of modifications, some of which you will explore in the homework. In practice, of course, the objective is to model enough features to capture the main features of the system, while keeping the system simple enough to study and extract some information.

# 5 Graphing solutions to systems; the phase plane

There are two main ways of graphing solutions to systems of differential equations. Each conveys the information of the solution in a slightly different way. I will illustrate both using the Romeo and Juliet example from section 2.

## 5.1 Graphing with respect to $t$

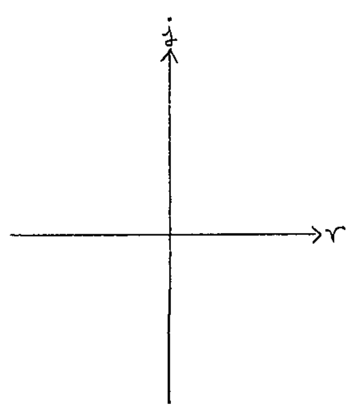
The first method is simply to graph both functions on the same axes, with the horizontal axis corresponding to the time $t$. For example, the solution to the Romeo and Juliet system, with initial condition $r(0) = 1, j(0) = 0$ is $r(t) = \cos t,\ j(t) = -\sin t$. So the two functions, graphed together, appear as follows.




Here the horizontal axis is the variable $t$, the blue curve shows $r(t)$, and the red curve shows $j(t)$. So from here we can see the behavior of the system: each person's interest in the other oscillates back and forth endlessly, one slightly out of phase with the other.

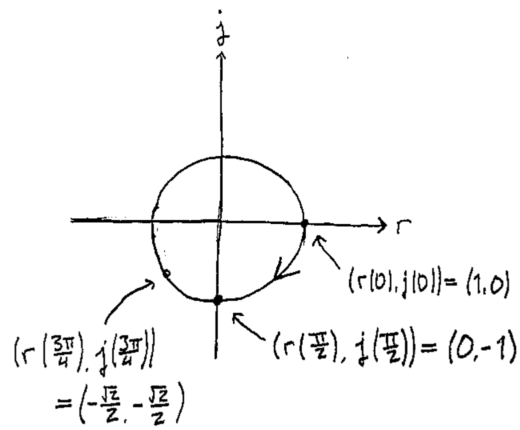
## 5.2 Graphing on the phase plane

The second method, which conveys slightly less information but is much more useful for many purposes, makes use of the so-called *phase plane*. In this setting, we draw a plane whose axes are $r$ and $j$.

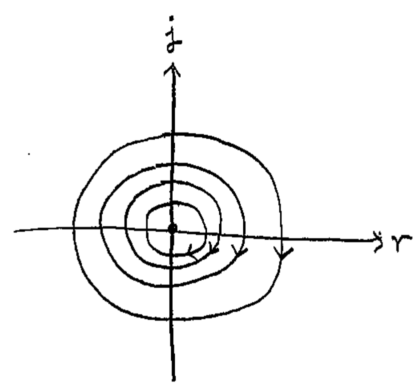



The plane is called the *phase plane*, and we will be working with it extensively over the final two classes. The key fact about the phase plane is that *points in the phase plane correspond to states of the system.* Put another way, a point in the phase plane describes precisely how Romeo and Juliet could feel at one particular moment. The question is: how does one such state move into another?

To plot one particular solution, we simply vary $t$, and plot the curve swept out by the points $(r(t), j(t))$. You may have seen this previously as parametric graphing. In the case of beginning at $r(0) = 1, j(0)$, this is what this trajectory looks like, roughly. Several points are marked on it to illustrate that the points on this trajectory correspond to states of the system at different moments in time.




One benefit of the phase plane is that it makes it easy to plot multiple trajectories at once. Different initial conditions will lead to different paths of the solution through phase space. The various solutions to the system will essential chop phase space up into many trajectories. For example, here are a number of possible trajectories for the Romeo and Juliet system. In fact, all of them will be circular orbits around the origin.




In fact, there is one trajectory that does not move at all: the *equilibrium* $r = 0, j = 0$ at the very center corresponds to a situation where $r(t)$ and $j(t)$ remain constant for all time. Notice that the equilibrium is enclosed by tightening circles; it wouldn't have any place to go even if it did want to evolve over time.

# 6 Second-order equations as systems

It turns out that second-order homogeneous equations can be regarded as systems of first-order equations. To see this, consider the standard form for such a differential equation.

$$x''(t) + bx'(t) + cx(t) = 0$$

To make this into a system, just define a function $y(t)$ to be $y(t) = x'(t)$. Then the differential equation can simply be written as $y'(t) = -by(t) - cx(t)$. In fact, we have a system of differential equations, for the *pair* of functions $x(t), y(t)$.

$$\begin{cases} x' = y \\ y' = -cx - by \end{cases}$$

Indeed, these are totally equivalent points of view, and for many purposes a very good way to understand a second-order equation is by turning it into a system of first-order equations. Notice, for example that an initial condition for this system consists of values for $x(0)$ and $y(0)$, i.e. for $x(0)$ and $x'(0)$, which was the same sort of initial condition as was needed before.

# 7 Apprendix: Systems of linear homogeneous equations

This appendix gives a brief glimpse of where this subject would be in a more advanced course (such as 21B). Unfortunately, it would be difficult to do any justice to this topic in such a short appendix, so this will only be a rough overview with unsupported claims.

A common sort of system of differential equation is a *linear* system. Here is a simple example.

$$\begin{cases} x' = 2x - 3y \\ y' = 3x - 5y \end{cases}$$

In general, a linear homogeneous system will look like the following. Here, $a, b, c, d$ are constants.

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

As the previous section showed, a second-order homogeneous equation can be transformed into such a system. In fact, there is a deep analogy between second-order equations and systems like these. In fact, systems also have a characteristic equation, which is given as follows.

$$r^2 - (a + d)r + (ad - bc) = 0$$

You can verify that in case the system comes from a second-order homogeneous equation, this is the same as the characteristic equation that we studied in that context.

Just like with second-order homogeneous equations, the characteristic polynomial determines the behavior of the solutions: if it has two real roots, then the general solution is build out of exponentials; if it has to imaginary roots, then the general solution is build out of sines and cosines; if it has two complex roots, then it is built out of exponentials multiplied by sines and cosines.

The same sort of picture persists for linear systems of differential equations that relate any number of functions: there is a characteristic equation, and whether or not its solutions are real determine whether the main behavior is exponential or oscillatory.