# Lecture 27: Autonomous equations

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## 1 Introduction

This lecture is concerned with qualitative analysis of a particular family of differential equations, called autonomous differential equations. Autonomous equations describe situations in which a particular quantity (such as the size of a population) grows and shrinks according to a model that only depends on the quantity itself. These tend to be very simplified models; due to their simplicity, they are fairly easy to study and understand. This lecture concerns only the qualitative study of such equations, while the next lecture will describe a way to solve them explicitly.

There will be no class on Friday due to Veteran's day.

The reading for today is Gottlieb §31.3. The homework is problem set 26. You should begin working on weekly problems 26 and 27.

## 2 What is an autonomous equation?

In simple terms, an autonomous differential equation describes a situation where the rate of change of a quantity is determined entirely by the quantity itself, and not by any external factors (such as the time or other quantities). For example, here are some examples that we have encountered that are modeled by autonomous equations.

- The population of rabbits on an island with unlimited resources and no predators. This is autonomous because the growth rate depends only on how many rabbits are available to breed.
- The population of rabbits on an island with unlimited resources but a single predator which eats the same number of rabbits each year.
- The number of people in an isolated village of 5000 people that have heard a rumor. This is autonomous because the spread rate of the rumor depends only on how many people have heard the rumor (and also how many people have not heard it, but since the population is constant this also depends on the number of people who have heard the rumor).
- The amount of money in a bank account that collect 3% per year nominal interest (compounded continuously) and has 1000 dollars withdrawn from it per year (at a continuous rate). This is autonomous because the rate of change of the amount of money in the account depends only on the current amount of money in the account.

By contrast, here are similar situations which are *not* autonomous.

- The population of rabbits on an island whose resources are slowly diminishing due to global warming. This is not autonomous, because the resources available to the rabbits vary with time.
- The population of rabbits on an island in the presence of a fluctuating population of predators. This is not autonomous because the growth rate of the rabbit population will depend not just on the rabbit population, but also on the predator population.

- The number of people in a non-isolated village that have heard a rumor. This is not autonomous because the total number of people in the village may be varying with time, so the spread of the rumor will depend not just on the number of people who have heard it, but also the total number of people in the village.
- The value of the assets in an investment portfolio. The rate of growth of the portfolio will naturally depend not just on the amount invested, but also with market conditions (which vary with time). So this situation is also not autonomous.

*Observation* 2.1. From the above examples, you might observe that the autonomous models are generally simple and naive models, while the non-autonomous models take more detailed circumstances into account. This is often the case; an autonomous model might be chosen for its simplicity, at the expense of its representational power. On the other hand, more complicated models can often be made autonomous by modeling the behavior of a *larger system:* e.g. in the second example above, the system of (rabbit population, predator population) might be given an autonomous model, where what we are modeling is a pair of interacting quantities, not just a single quantity. We will consider this type of situation beginning in lecture 32.

## 2.1 Autonomous equations, more formally

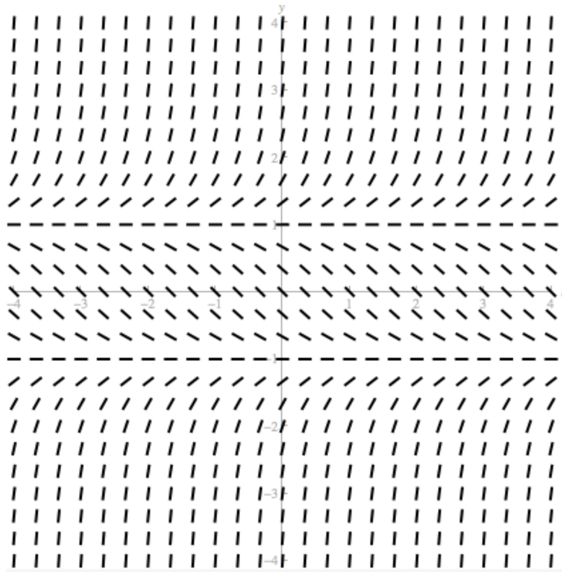
In mathematical terms, an autonomous first-order differential equation is an equation of the following form.

$$\frac{dy}{dt} = f(y)$$

This encodes the informal description given above: the rate of change of the quantity $y$ depends only on the quantity itself. For example, here are how the four examples in the previous subsection might be modeled as autonomous equations.

$$\begin{aligned} \text{Rabbits} \quad & \frac{dy}{dt} = Cy \text{ (where } C \text{ is a constant)} \\ \text{Rabbits with a predator} \quad & \frac{dy}{dt} = Cy - D \text{ (where } C, D \text{ are constants)} \\ \text{The spread of a rumor among 5000 people} \quad & \frac{dy}{dt} = Cy(5000 - y) \text{ (where } C \text{ is a constant)} \\ \text{A bank account with 3\% interest and 1000 in annual withdrawals} \quad & \frac{dy}{dt} = 0.03y - 100 \end{aligned}$$

Autonomous equations are easy to understand in terms of their corresponding slope field. For example, here is a slope field for the autonomous equation $\frac{dy}{dt} = y^2 - 1$.

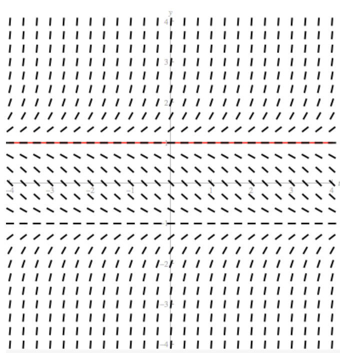


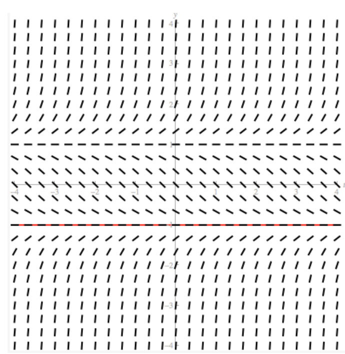
The fact that this equation is autonomous is reflected in the fact that the slopes of the dashes do not depend on the value of $t$, but only on the value of $y$. In particular, all of the "vertical slices" look the same.

# 3 Equilibrium solutions

An *equilibrium solution* of an autonomous differential equation is simply a solution function which is constant. In other words, it is a value for the quantity being modeled which causes the rate of change of the quantity to be 0. Therefore, according to the model, the quantity will not change.

Given a slope field, it is easy to spot the equilibrium solutions: they are the places where the dashes are perfectly horizontal.





Generally, the equilibrium solutions will correspond to some sort of equilibrium in the situation being modeled. For example, they might correspond to a population at a stable level, based on the resources and predators in the environment.

*Example* 3.1. Suppose that the number $R(t)$ of people in a town of 5000 who have heard a rumor is modeled by $R'(t) = R(t)(5000 - R(t))$. Then the equilibrium solutions are $R(t) = 0$ and $R(t) = 5000$, since these are the values for which $R'(t) = 0$. Qualitatively, these correspond to the situations where no one has heard the rumor and where everyone has heard the rumor.

# 4 Representative families of solutions

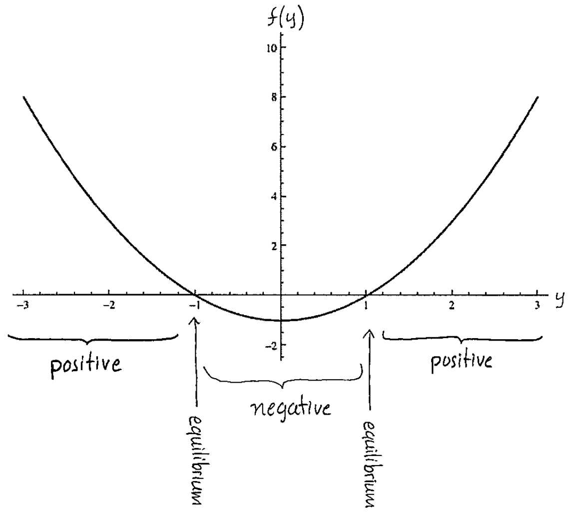
In addition to the equilibrium solutions of an autonomous equation, there are two other situations that might occur.

- The quantity could be increasing.
- The quantity could be decreasing.

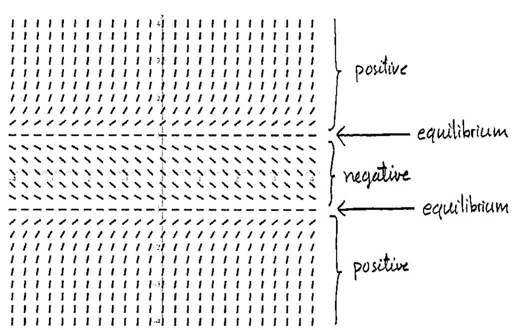
For example, consider the following autonomous equation.

$$\frac{dy}{dt} = y^2 - 1$$

Let $f(y) = y^2 - 1$. We wish to know which values of $y$ will makes this function positive or negative. Below is a plot of the graph of this function, with the places where it is positive, negative, and 0 marked.



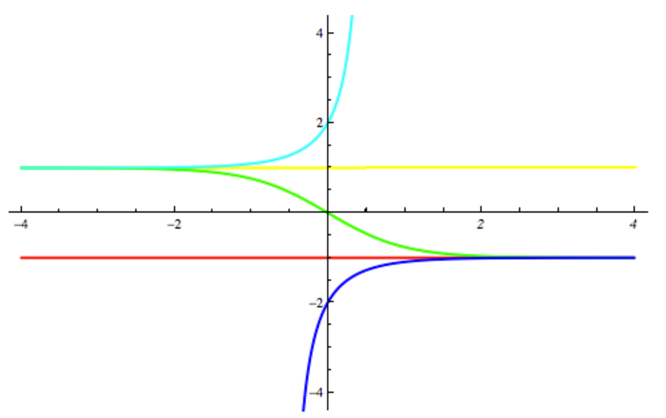

Corresponding to these values of $y$, we have regions of the slope field of the equation $y' = y^2 - 1$ where the dashes have positive slope, negative slope, or zero slope (equilibrium).




Now, by choosing one initial condition for each of these segments (increasing and decreasing blocks), we obtain what is called a representative family of solutions. For example, we could consider the following five initial conditions. These are selected to include both equilibrium solutions (1 and $-1$) and also one point in each of the intervals in between.

$$
\begin{aligned}
y(0) &= -2 \\
y(0) &= -1 \\
y(0) &= 0 \\
y(0) &= 1 \\
y(0) &= 2
\end{aligned}
$$

The following plot shows the five resulting solution curves.

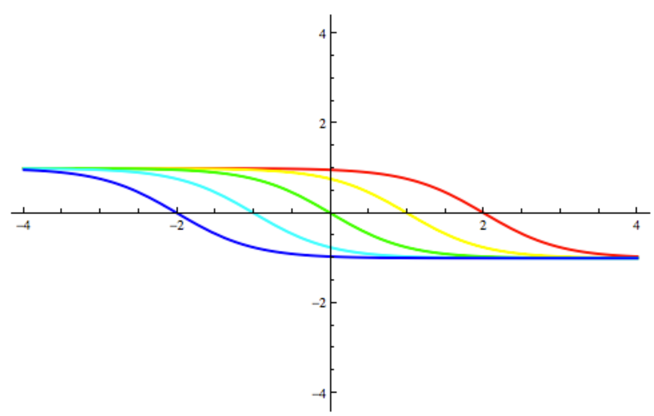



These five solutions are called a *family of representative solutions*. In general, a set of solutions will be called a family of representative solutions if:

- It includes all equilibrium solutions.
- It includes a solution between each pair of equilibrium solutions.

Such family are called representation for the following reason: *each solution of an autonomous differential equation is either an equilibrium solution or a horizontal translate of some solution from a family of representative solutions.*

To illustrate this point, here are several other solution curves lying between the two equilibrium solutions. Notic that they are all simply horizontal translates of each other.



Solution curves above $y = 1$ or below $y = -1$ can be obtained in the same way.

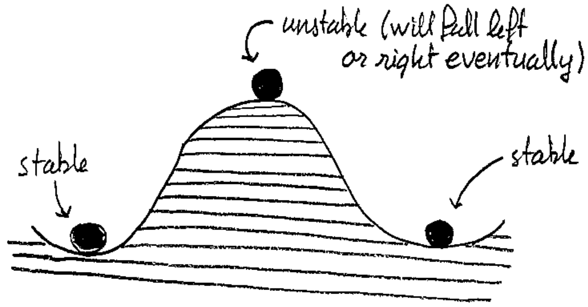
Notice that for very large or very small values of $t$, all of these graphs cling to one equilibrium or the other and become essentially indistinguishable.

## 5 Stability of solutions

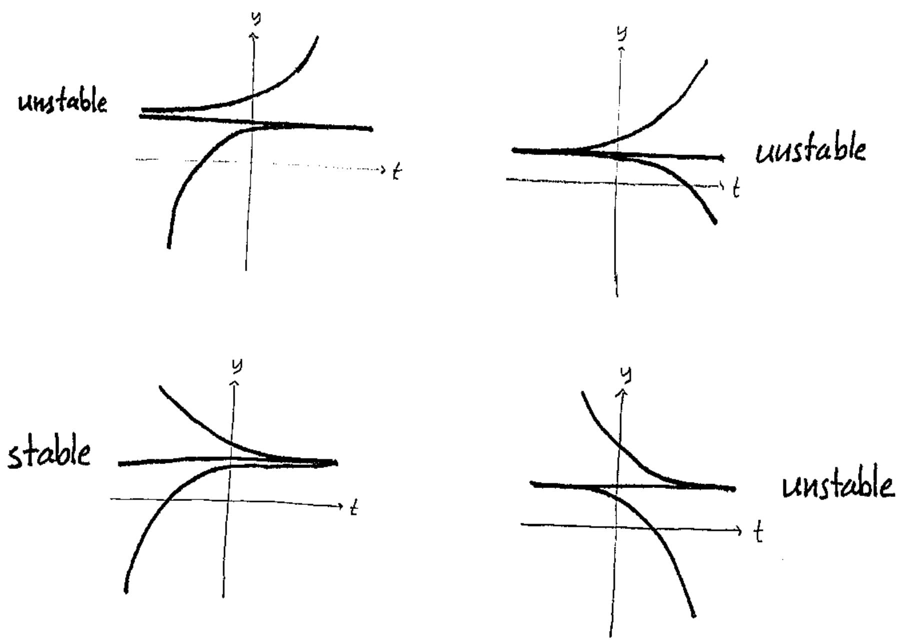
Look again at the five representative solutions to $y' = y^2 - 1$ on the previous page. The behavior of all solutions revolves around the equilibrium solutions $y = -1$ and $y = 1$. But these two equilibria are qualitatively different: the equilibrium $y = 1$ seem to *repel* nearby solution curves away from itself, while the equilibrium $y = -1$ seems to *attract* nearby solution curves to itself. For this reason, we say that $y = 1$ is an *unstable equilibrium*, while $y = -1$ is a *stable equilibrium*.

In general, an equilibrium solution to an autonomous equation is called *stable* if solution curves on either side of it converge to it for large $t$, and it is called *unstable* otherwise.

To explain the terminology, think of an unstable equilibrium as a value of $y$ that is precariously balanced, but such that even a small perturbation could cause it is explode away from the equilibrium. The solution $y = 1$ is certainly such a situation for the equation $y' = y^2 - 1$. Practically speaking, a physical system will never remain in an unstable equilibrium for long; even if the relevant quantity is precisely balanced at the equilibrium, sooner or later something will ever so slightly perturb the system and cause the quantity to flow towards one of the stable equilibria. You could visualize this by thinking of a ball rolling along terrain with peaks an valleys (as illustrated below).




It is easy to recognize stable and unstable equilibria from a representative family of solutions. Four examples are shown below, of equilibria with nearby solution curves.




Note that the two cases in the upper-left and lower-right are sometimes called "semi-stable" equilibria, since solution curves approach them from one side by not the other. However, for our purposes, we will regard "semi-stable" equilibria as being unstable as well.

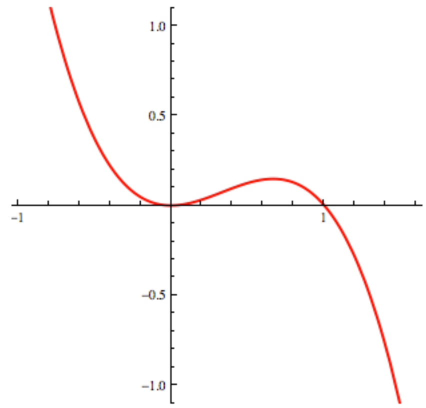
## 6 Qualitative analysis

Given an autonomous differential equation $\frac{dy}{dt} = f(y)$, the proces of *qualitative analysis* refers to the following sequence of steps.

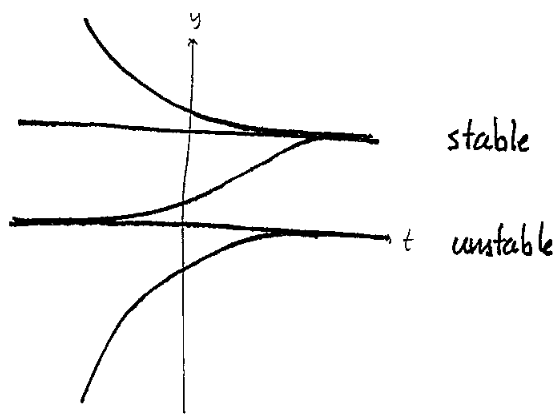
1. Identify all equilibrium solutions (i.e. values of $y$ such that $f(y) = 0$).
2. Determine whether $\frac{dy}{dt}$ is positive or negative in between each pair of equilibria, as well as above the largest and below the smallest equilibrium.
3. Sketch a representative family of solutions for each of these cases, increasing or decreasing as appropriate.
4. Identify each equilibrium as stable or unstable, based on whether nearby solutions approach it for large $t$ or not.

Once this process has been completed, we essentially understand the key points of the behavior of the system modeled by this differential equation, since all solutions not in the representative family can be obtained by horizontally translating members of the representative family.

*Example* 6.1. Apply qualitative analysis to the differential equation $\frac{dy}{dt} = y^2 - y^3$.
First, we must find the equilibrium solution. By factoring, $y^2 - y^3 = y^2(1-y)$, so this is equal to 0 for $y = 0$ and $y = 1$. So these are the two equilibrium solutions. The graph of the function $f(y) = y^2 - y^3$ is shown below.




From this picture, it is apparent that solutions below $y = 0$ are increasing, solutions between $y = 0$ and $y = 1$ are also increasing, and solutions above $y = 1$ are decreasing. Hence here is a sketch of a representative family of solutions.




Since nearby curves approach it from either the top or the bottom, $y = 1$ is a stable equilibrium. Since solutions peel away from it from above, $y = 0$ is an unstable equilibrium.

## 7 Example: population growth with a carrying capacity

Here is a standard example of modeling using differential equations. Again, suppose that there is a population of rabbits on an island (although we could just as easily be modeling many other similar situations), but that the resources are finite. In particular, there is enough food on the island to sustain a population of 3000 rabbits. Let $R(t)$ be the number of rabbits on the island after $t$ years.

A measure of how quickly the population might grow or shrink might be given by $3000 - R(t)$, since this measures how much excess food there is (or, if $R(t) > 3000$, what the deficiency is). Thus if $3000 - R(t)$ is positive, then there is an abundance of food, while if it is negative, then there is a shortage.

Now, supposing that a food surplus or shortage will affect each rabbit roughly equally, the rate at which rabbit are dying off or being born should be proportional both to the level of surplus or famine, as well as the number of rabbits (since more rabbits means more rabbits to breed, but also more to die off from shortage of food). This suggests the following model.

$$R'(t) = C \cdot R(t)\,(3000 - R(t))$$

Here $C$ is just a constant of proportionality.

Differential equations like this are called *logistic equations*, and their solutions are called *logistic functions*. The number 3000 here is called the *carrying capacity*, since it is the tipping point between a growing and a shrinking population.

Note that we have considered the same sort of model before, in considering the spread of a rumor in a small population. Indeed, logistic equations occur in many different contexts.

Now, the equilibrate here are $R(t) = 0$ and $R(t) = 3000$. That is, the rabbit population will be stable if it is at carrying capacity or if there are no rabbits at all.

Furthermore, since $CR(t)(3000-R(t))$ will be negative when $R(t) > 3000$, positive when $0 < R(t) < 3000$, and negative when $R(t) < 0$, we see that 0 is an unstable equilibrium and 3000 is a stable equilibrium. In terms of the situation being modeled: the population will tend to 3000 rabbits over time and remain fairly stable at that level. On the other hand, a population of 0 rabbits is stable, but the introduction of even a couple rabbits will break this stability and quickly lead to an exploding population (which will slow down as it reaches 3000). So in this sense, 0 is an unstable equilibrium.

Note that of course many other models could be used for a population in a similar situation that might lead to different results. For example, it is possible that a population needs a certain baseline number in order not to go extinct due to lack of biodiversity. In this case, 0 would be an stable equilibrium. In this case, there might be another quantity, the *minimal viable population*, such that a population which falls below this level will go extinct, while a population just above this level will continue to grow until reaching carrying capacity. Thus we might expect the minimal viable population to be a unstable equilibrium.