Lecture 26: Slope fields

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1 Introduction

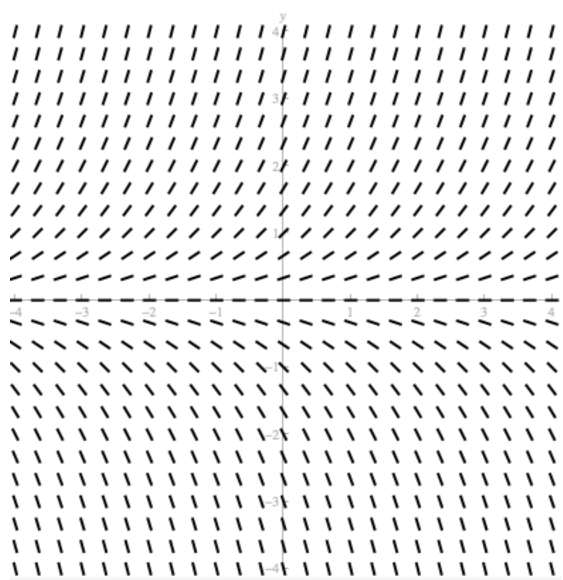
When faced with a differential equation, it is often useful to form a general impression of the shape of its solutions. The technique of slope fields is a simple visualization technique, which often reveals the main behavior of the solutions, and how different solutions of the same equation differ from each other.

Slope fields typically are used for differently equations of the form $y'(t) = f(y, t)$, where f is some function of two variables. The method also provides some intuition for why solutions to such differential equations always exist and are unique, at least when f is reasonably "nice" (e.g. continuous). This is briefly discussed in section 3.

The reading for today is Gottlieb §31.2. The homework is problem set 25. You should begin working on weekly problem 25.

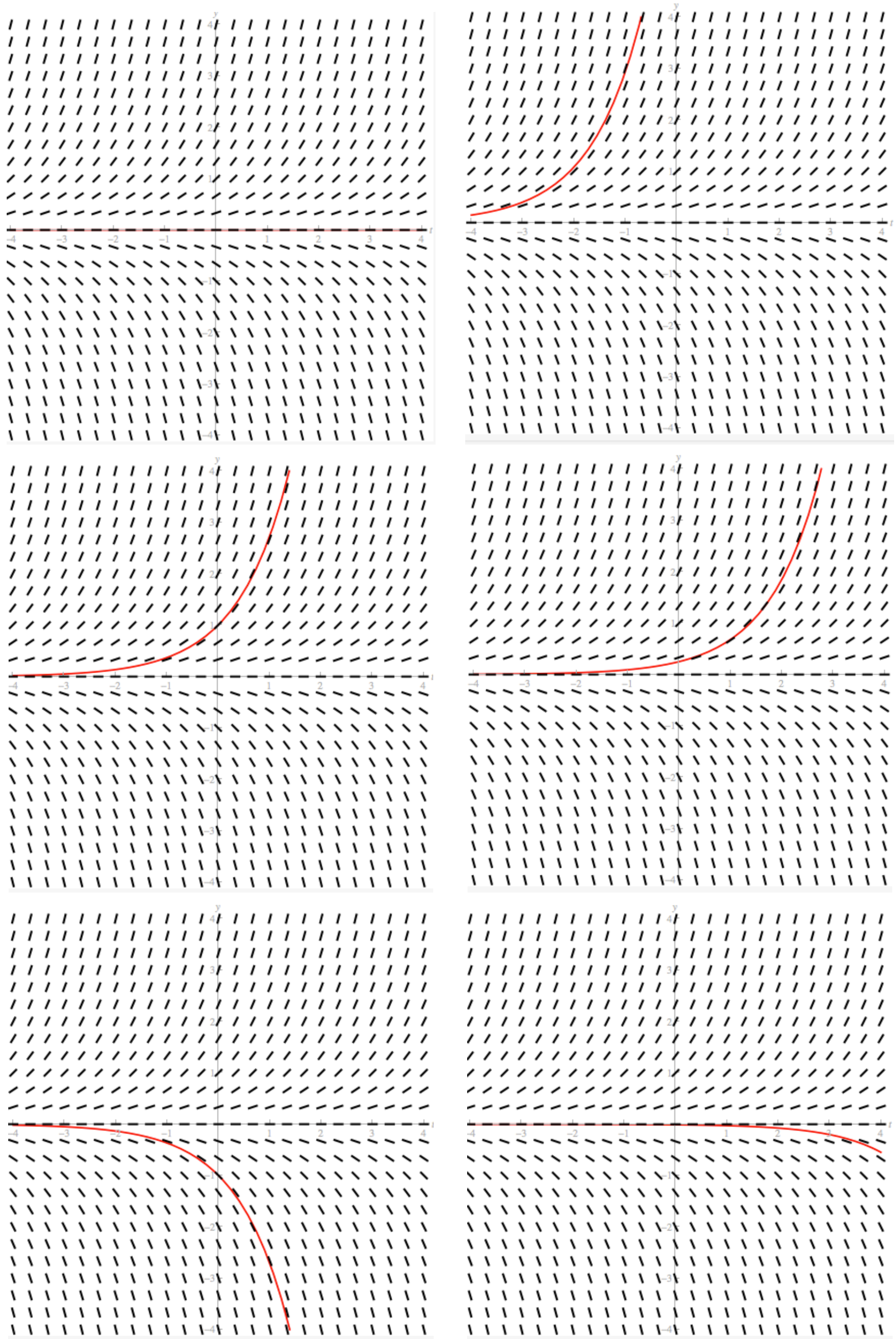
2 Slope fields

Consider a simple differential equation: $y' = y$. Regarding y as a function of x, what would the graph of this function have to look like? One thing we can do to try to see this information is to notice that if we know the value of y at some point, then we also know what the slop of the function needs to be at that point. We can record this slope in a picture with a small line segment. Drawing such line segments at many points, we obtain a picture that is fairly useful in "seeing" this differential equation.



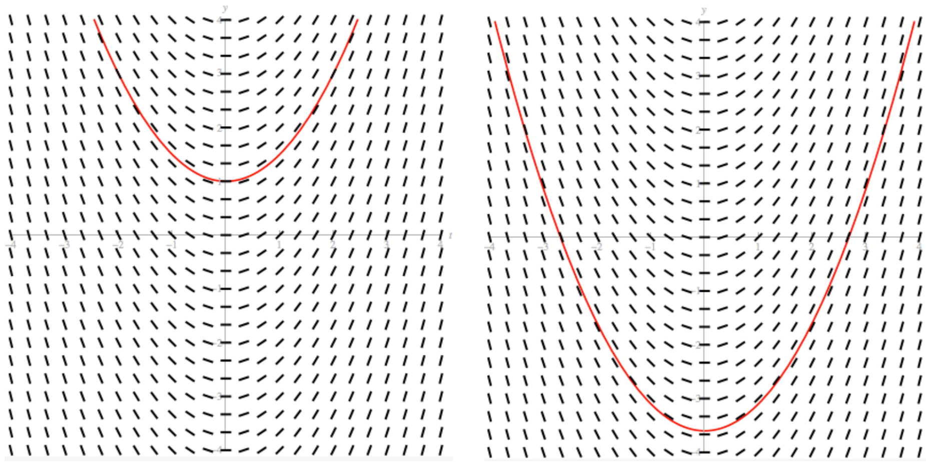
To explain this picture: the small line segments show roughly what the graph of a solution must look like near the dash, since the dash has the same slope that a solution function would have to have. In particular, where $y = 0$, the segments are totally flat. As y grows, they become steeper; as y shrinks, they slant more and more downward.

To see how this picture can shed light on solutions of the equation, note that we could approximate a solution to the equation by "following" the dashes, so that we draw a curve that is parallel to nearby dashes. To illustrate what is meant by this, consider the following graphs drawn over this slope field.



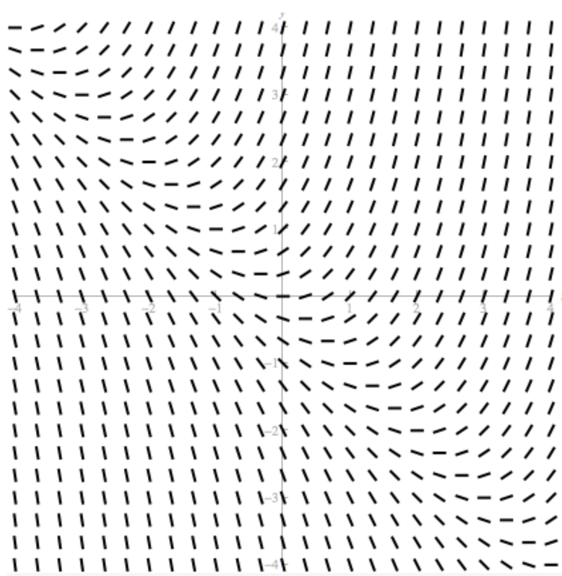
In fact, the curves drawn are solutions to this differential equation. Notice that we could guess their general shape from the "flow" of the slope field. It is also worth noting that in this case, the different solution curves can often be obtained by translating other solution curves left and right.

Consider another seemingly similar differential equation, with rather different behavior: $y' = x$. The slope field and two possible solution curves are shown below. In this case, the dashes are totally flat along the y axis, and get steeper moving left or right. Essentially, this is because the right side of the equation $y' = y$ depends only on y, not x.

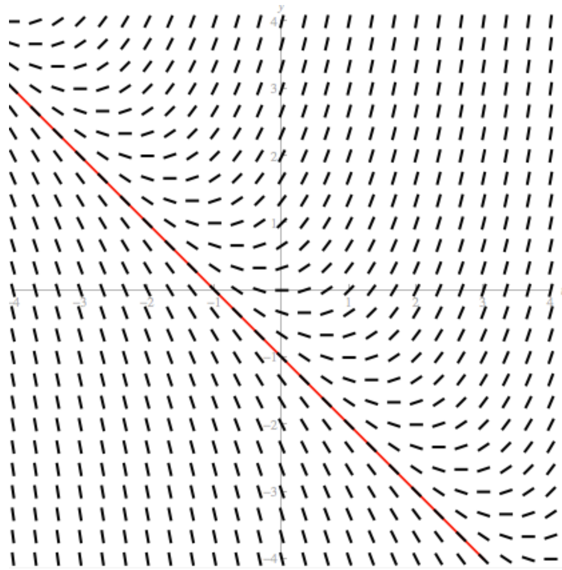


Notice that in this case, translating a solution curve up or down results in another solution curve. This is because the right side of the equation depends only on x, not y.

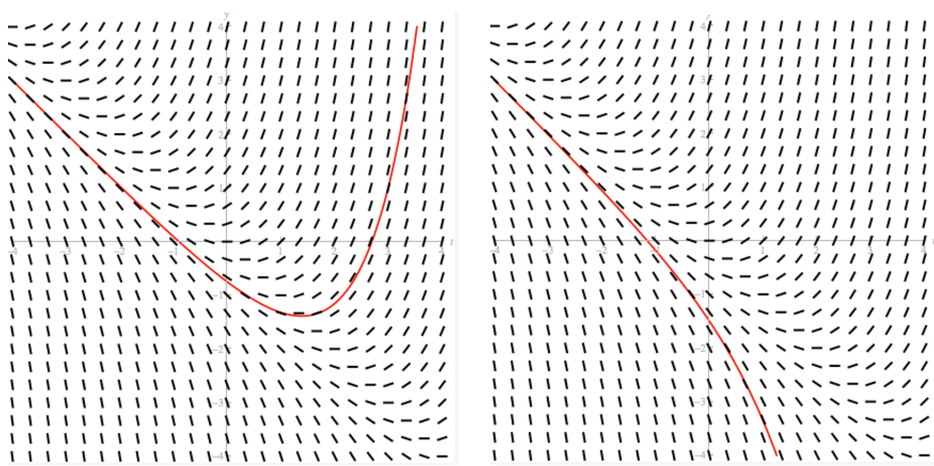
Now let's consider a slightly more interesting example, where the right side of the equation depends on both x and y. Consider the equation $y' = x + y$. The slope field looks like this.



Unlike in the previous two cases, we see some qualitatively different behavior in some different cases here. For example, it looks like there is a fairly straight "spine" running down and to the right, giving one rather simple solution curve.

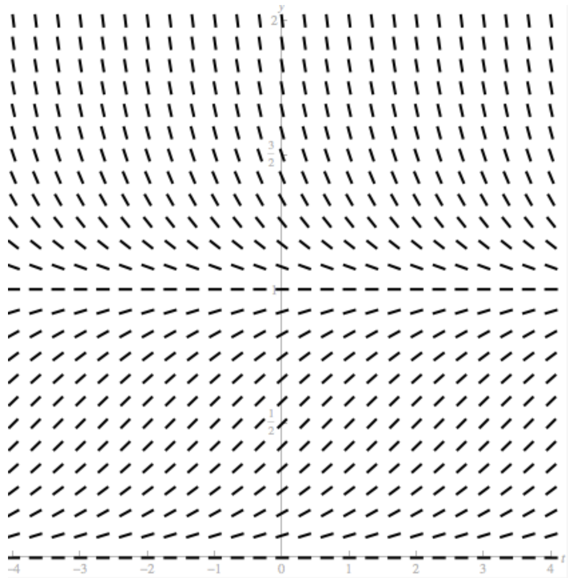


Above and below this spine, we see some other behavior: solutions which follow the spine and first, and then either bounce upwards or spike downward.

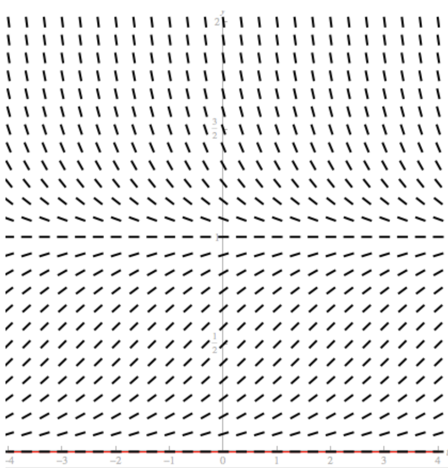


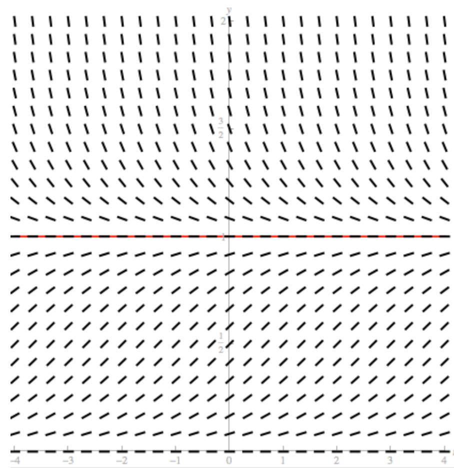
This sort of behavior – one very simple "dividing line" off of which other solutions bifurcate to one side or the other – is very common in differential equations. Much of the qualitative aspect of the subject focuses on finding these solutions, and understanding how exactly the other solutions feel off of them.

Probably the most iconic example of this way of thinking is the so-called "logistic" equation $y' = y(1-y)$. We have encountered equation of this form (with different constants, but essentially the same shape) in considering spread of disease or rumor in an isolated population, or to model the growth of an animal population in the presence of scare resources. Here is the slope field for this equation.

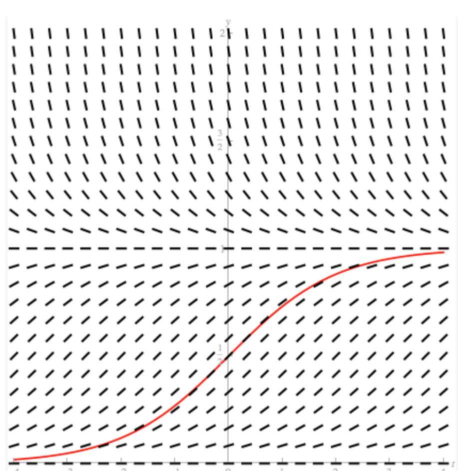


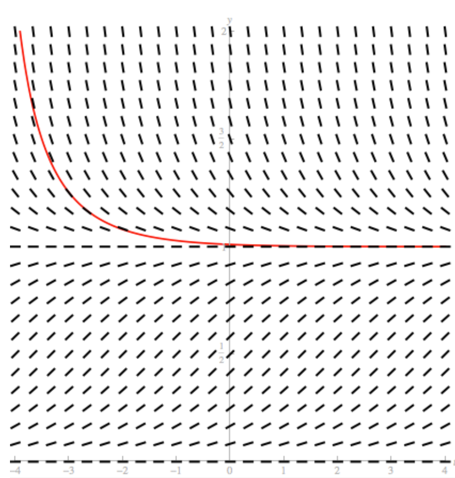
Here, there are two "straight line" solutions. Later, we will refer to these as *equilibrium solutions*.





The other solutions essentially "cling" to these two solutions, either for large t or small t.





In terms of population dynamics, the equilibrium solutions correspond to an extinct population (0) or a population at carrying capacity. The slope field technique allows the interplay between these notions to be easily visible.

3 Uniqueness of solutions

Notice that if we give a complete "initial value problem" like the following, then it appears from looking at the slop field that there should be one, and only one, solution curve that matches it.

$$\begin{aligned} y' &= y(1-y) \\ y(0) &= 0.1 \end{aligned}$$

Physically speaking, this makes sense: given a model for a population, for example, the population at all times should follow deterministically from the initial state.

In terms of slope fields, the initial condition correspond to picking a single point on the slope field. The solution should then correspond to "following the grain" of the slope field. Indeed, it appears that this should always be possible, and that there should be one *and only one* solution. This is the basic idea behind the following informal statement: *a differential equation $y' = f(x, y)$, where f is a well-behaved function, has a unique solution for any initial condition.* Of course, for this to be a theorem, I have to tell you what I mean by "well-behaved." Let it suffice to say that almost any function we will deal with in this course qualifies. The textbook has a precise statement of a version of this theorem on page 994. Results of this type are generally referred to as "existence and uniqueness theorems."