Lecture 22: Power series I

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1 Introduction

We have nearly finished discussing sums of infinite series. In this lecture and the next, we will consider *power series*, which are essentially series expressed in terms of a variable (usually x). Taylor series are really the only example we care about. In fact, you can basically equate "power series" and "Taylor series" in your mind; the only difference is that a power series is defined directly as a series, whereas a Taylor series is always described as coming from some function and some point.

This lecture defines power series and considers their convergence properties. For a power series, we no longer speak of it converging or diverging per se, but we rather consider *which values of x* cause it to converge or diverge. The basic notions here are the interval of convergence and radius of convergence. We will describe these notions today, and then go into more detail about the connection between power series and functions in the next class.

The reading for today is Gottlieb §30.3, beginning at "Power Series" on the bottom of page 944, as well as the "Power series" handout (under "Reading for the course"). The homework is problem set 21 (this includes weekly problems 22 and 23, even though 23 is inadvertently omitted from the problem sheet) and a topic outline.

2 The notion of a power series

A power series is something like one of the following.

$$1 + x + x^2 + x^3 + \cdots$$
$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$
$$1 + (x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \cdots$$

All of these are essentially infinite series, except that each term is expressed in terms of a variable x, not directly as a number. In fact, the terms have a simple form: they are each a constant times some power of x or $(x-c)$ (where c is a constant). The number c, like for Taylor series, is called the *center*.

In general, a power series will have the following form.

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

Here $a_0, a_1, a_2, \cdots$ are all constants (the *coefficients*), as is c (the *center*).

Given a power series, we can pick any value of the variable x and substitute it into each term. The result will be a series in the usual sense. For example, the following three infinite series are obtained from the power series above by using the value $x = 3$.

$$1 + 3 + 3^2 + 3^3 + \cdots$$
$$x - \frac{1}{2}3^2 + \frac{1}{3}3^3 - \frac{1}{4}3^4 + \cdots$$
$$1 + (3-1) + \frac{1}{2!}(3-1)^2 + \frac{1}{3!}(3-1)^3 + \cdots$$

All three of these are now series in the usual sense: infinite sums of *numbers.* Like any other series, these might converge or might diverge. In fact, two of the three series above diverge (which two?).

Since a power series does not become a series of numbers until we choose an actual value for x, we do not speak of a power series converging or diverging in itself. Instead, we consider *which values of x* will make the series converge, and which will not.

3 The radius of convergence

It turns out that, for a power series with center c, the values of x which cause the series to converge are arranged around the center. In fact, there is a number (or, possibly, ∞), called the *radius of convergence*, which tells how far x must be from the center before the series diverges.

Theorem 3.1. *Consider a power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$. *There is a quantity R (either a nonnegative number or ∞) such that the following holds.*

- *Whenever a value of x is chosen with $|x-c| < R$, the power series converges absolutely.*
- *Whenever a value of x is chosen with $|x-c| > R$, the power series diverges.*

The quantity R is called the radius of convergence.

To see where the radius R comes from, note that we can apply the ratio test to a power series, for any chosen value of x.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Now, if this limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then the radius R will simply be $\frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$. If this limit is 0, then $R = \infty$. If this limit diverges to $+\infty$, then $R = 0$.

Now, observe the following two facts.

- If $R = \infty$, then the power series converges for all values of x.
- If $0 < R < \infty$, then the series convenes absolutely for x in $(c - R, +R)$, and diverges for $|x - c| > R$, but may converge or diverge at $x = c - R$ and $x = c + R$ (where the ratio test is inconclusive).
- If $R = 0$, then the power series converges only at $x = c$, and nowhere else.

Example 3.2. Consider the power series $\sum_{n=0}^{\infty} x^n$. This converges absolutely whenever $|x| < 1$. The radius of convergence, therefore, is $R = 1$.

Example 3.3. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then since $\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$, the radius of convergence is $R = \infty$. This power series converges for all values of x (to e^x, in fact).

Example 3.4. Consider the power series $\sum_{n=0}^{\infty} n!x^n$. Then since $\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n = \infty$, the radius of convergence is $R = 0$. This series only converges for the single value $x = 0$.

4 The interval of convergence

If the radius of convergence is $R = 0$ or $R = \infty$, then we know everything about the convergence of a power series. However, if $0 < R < \infty$, there are two more values that are unknown, namely $c - R$ and $c + R$. If we can determine in an *ad hoc* way whether the series converges at these values, then we will know the complete set of values for which the power series converges. This set is called the *interval of convergence*.

Example 4.1. What is the interval of convergence of the power series $\sum_{n=0}^{\infty} x^n$?

The radius of convergence is $R = 1$, and the center is $c = 0$. So the only two remaining values to check are $x = -1$ and $x = 1$. Now, from studying geometric series we know that this series diverges for both of these values. Therefore the interval of convergence of this power series is $(-1, 1)$.

Example 4.2. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$? Using the ratio test, we see that this series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. So the radius of convergence is $R = 1$, and the two remaining values to check are $x = -1$ and $x = 1$. These give the following two series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

These are the familiar harmonic and alternating harmonic series. The first converges, but the second diverges. So this power series converges at $x = -1$ and diverges at $x = 1$. So the interval of convergence is $[-1, 1)$.

5 Manipulation of power series

One of the strengths of power series is that they can be manipulated term by term in various ways. There are two ways that are worth mentions today (we will discuss more next time).

The first manipulating is multiplying by constants or powers of $(x - c)$. This does not change the radius of convergence, since multiplying by constants will never do that.

Example 5.1. We know that for any x, e^x is equal to the value of the power series $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, whose radius of convergence is infinite. Then by multiplying by x, we see that the function xe^x has a similar representation by a power series.

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3!}x^4 + \cdots$$

The second manipulation is substitution. Any expression may be substituted for x to obtain a new series.

Example 5.2. How would you find a power series expansion for e^{x^2}? Simply substitute x^2 for x in the Taylor series of e^x.

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6 + \cdots$$

This will converge for all values of x^2, i.e. for all values of x.

Example 5.3. How would you find a power series expansion for $\frac{1}{1+2x}$? Well, recall that we have a power series expansion for $\frac{1}{1-x}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{1}$$

This converges for x in $(-1, 1)$. Therefore, substituting $-2x$ for x, we obtain the following.

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = 1 - 2x + 4x^3 - 8x^3 + \cdots \tag{2}$$

Now, since the series in equation 1 converges for x in $(-1, 1)$ and the equation holds in these cases, the series in 2 converges for $-2x$ in $(-1, 1)$, and the equation holds in these cases. In other words, the interval of convergence of the series is $(-\frac{1}{2}, \frac{1}{2})$.