Lecture 21: The ratio test

Nathan Pflueger

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1 Introduction

Geometric series are the emblematic infinite series: they are easy to sum explicitly, and it is easy to determine whether they converge. For this reason, they are often used as benchmarks to which other series can be compared.

The basic result that can be used to compare series to geometric series is the ratio test. Essentially it generalizes the basic fact about geometric series (they converge as long as the ratio has magnitude less than 1) to series where the ratio of successive terms is not constant, but does approach some limit.

The ratio test is especially important when considering Taylor series. We will briefly consider this connection today, and elaborate on it later.

The reading for today is Gottlieb §30.5, starting from the bottom of page 972. The homework is problem set 20 (which includes weekly problems 20 and 21) and a topic outline.

2 The statement of the ratio test

Consider a series $\sum_{n=1}^{\infty} a_n$. Recall that this series is called geometric if all ratios of consecutive terms are the same.

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots$$

Furthermore, in this case, then call this constant value the *common ratio*, and denote it by r. Then the most basic fact about geometric series is that this series will converge if and only if $|r| < 1$.

What happens in case these successive ratios are not constant? Bearing in mind that only behavior at infinity matters, perhaps we don't need these ratios to be constant, but merely approach some constant, L. In fact, the ratio test states that in this case, almost exactly the same conclusion holds.

Theorem 2.1 (The ratio test). *Suppose that $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$ is a series, and that the ratios of successive terms converge to some limit, L.*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

Then:

- *If $|L| < 1$, then the series converges.*
- *If $|L| > 1$, then the series diverges.*

Note that if $|L| = 1$, then the ratio test is inconclusive.

I shall not provide a detailed proof here. Essentially, what makes the argument work is that if the limit of the ratios is L, then the terms of the series are eventually bounded by a geometric series of ratio $L + \epsilon$ or $L - \epsilon$ (where ϵ is a very small number), which will converge or diverge according to whether $|L| < 1$ or $|L| > 1$.

The reason that the test is inconclusive for $L = \pm 1$ is that this is right on the cusp between convergence and divergence; the actual behavior will depend on things like which direction the ratios approach L from, and how quickly.

In fact, by considering absolute values and the fact that absolute convergence implies convergence, we obtain a slightly stronger statement (which follows readily from the original statement).

Theorem 2.2 (Generalized ratio test)**.** *Suppose that* $a_1 + a_2 + a_3 + \cdots = \displaystyle\sum_{n=1}^{\infty} a_n$ *is a series, and that the ratios of the absolute values of successive terms converge to some limit,* L*.*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

Then:

- *If* $L < 1$*, then the series converges absolutely (and hence converges).*
- *If* $L > 1$*, then the series diverges.*

Note that the generalized ration test will always establish either absolute convergence, divergence, or be inconclusive. It will always be inconclusive when applied to a conditionally convergence series.

3 Examples

Example 3.1 (Geometric series)*.* This is a bit of a silly example. Consider the series $1 + r + r^2 + r^3 + \cdots$. Then this can be written as $\sum_{k=0}^{\infty} r^k$, and the ratio of any two consecutive terms is $\frac{r^{k+1}}{r^k} = r$. Therefore the limit of the ratios of consecutive terms is $\lim_{k\to\infty} \frac{r^{k+1}}{r^k} = r$. So the ratio test implies that if $|r| < 1$, this series converges, and if $|r| > 1$, then the series diverges. Of course, we already knew that (and it the basis of the proof for the ratio test).

Example 3.2 (p-series)*.* Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where p is some constant. Then the limit of ratios of consecutive terms is calculated as follows.

$$\lim_{k\to\infty} \frac{1/(k+1)^p}{1/k^p} = \lim_{k\to\infty} \left(\frac{k}{k+1} \right)^p = 1.$$

So the ratio test in this case is inconclusive (of course, we know for which values of p this series converges, by other means). You can't win them all.

Example 3.3*.* Consider the following series (a combination of sorts of p-series and geometric series): $\displaystyle\sum_{n=1}^{\infty} \frac{n^{27}}{2^n}$. The ratio of successive terms is the following.

$$\begin{aligned}\lim_{n\to\infty} \frac{(n+1)^{27}/2^{n+1}}{n^{27}/2^n} &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{27}\left(\frac{2^n}{2^{n+1}}\right) \\ &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{27} \cdot \frac{1}{2} \\ &= 1^{27} \cdot \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

Since $|\frac{1}{2}| < 1$, the ratio test implies that this series converges.

Note that we could have used the integral test to establish the convergence of this series, combined with prodigious integration by parts. Clearly, however, the ratio test was a much more straightforward option.

The last example demonstrates a general principle: the presence of exponentials in the terms of a series (here, $\frac{1}{2^n}$) will generally overpower other parts of the series (here, x^{27}) from the standpoint of convergence.

4 The ratio test applied to Taylor series

The main utility of the ratio test comes from what it gives when it is applied to Taylor series. Indeed, consider the Taylor series for $\ln|1+x|$.

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

The general term looks like $(-1)^{n+1}\frac{1}{n}x^n$. This essentially has a part that looks geometric (x^n) and a part that does not. The ratios of successive terms has a very simple limit in this case.

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+2}x^{n+1}/(n+1)}{(-1)^n x^n/n}\right| = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x|$$

Therefore the ratio test gives the following conclusion: the Taylor series for $\ln|1+x|$ converges absolutely for $|x| < 1$, and diverges for $|x| > 1$. The number 1 is sometimes called the *radius of convergence*, and we will discuss this notion in more detail next time.

Consider another Taylor series.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$$

Now, the limit of the ratios of successive terms is the following.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| &= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} \\ &= |x| \lim_{n\to\infty} \frac{1}{n+1} \\ &= |x| \cdot 0 \\ &= 0\end{aligned}$$

Notice again that the limit is some constant times $|x|$. In this case, the constant is 0, so in fact this series converges absolutely for *all* values x.

It will frequently happen that the limit of successive terms of a Taylor series will be $|x|$ times some constant factor. The constant factor will determine how wide a range of x will still permit convergence, i.e. what the radius of convergence will be.

5 The root test

There is another test, closely related the the ratio test, which is also sometimes used. It is somewhat less easy to apply in most cases, so we won't spend a lot of time on it. It operates on the same principle: imagine that the series is a geometric series eventually, and see what the limit would have to be. One way to access this is by ratios of successive terms; another way is by taking roots. The statement is as follows.

Theorem 5.1 (The root test)**.** *Suppose that $\sum_{n=1}^{\infty} a_n$ is a series, and suppose that the limit of the nth roots of the magnitudes of the terms converges to some value, L.*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

Then:

- *If $L < 1$, then the series converges absolutely (and hence converges).*
- *If $L > 1$, then the series diverges.*

Example 5.2. Consider what happens when you apply this test to a geometric series $a + ar + ar^2 + \cdots$.

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|ar^n|} &= \lim_{n\to\infty} \sqrt[n]{a} \cdot |r| \\ &= |r| \lim_{n\to\infty} \sqrt[n]{a} \\ &= |r| \end{aligned}$$

Here the last step follows because the nth roots of a positive constant approach 1 as n grows.

The conclusion, like with the ratio test, is again what we already knew: if $|r| < 1$, then the series converges (absolutely) and if $|r| > 1$, then the series diverges.