Lecture 18: Geometric series

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1 Introduction

Geometric series are the main example of infinite series. They provide an important class of examples for the notions of convergence and divergence. They are also very important to the study of Taylor series, since the problem of finding the values of x where a Taylor series convenes is usually solved by using the comparison test to compare to a geometric series.

The reading for today is Gottlieb §18.1 (on finite geometric sums) and §18.2 (on geometric series). The homework is problem set 17 and a topic outline. You should begin working on weekly problems $18, 19$, and 20.

Contrary to what was originally written in the notes from the previous lecture, *this* lecture will be the last lecture covered on the next midterm exam.

2 Definition of a geometric series and examples

We have already seen and considered two main examples of geometric series.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

At the time, I gave a rough argument in words for the fact that both of these series converge, and even gave their sums. We will do this again in a somewhat more careful way today.

Both these series can be described in the similar way: the terms of the first series result from taking powers of $\frac{1}{2}$, while the terms of the second series result from taking powers of $-\frac{1}{2}$. In other words: the ratio of any two consecutive terms of the first series is $\frac{1}{2}$, and the ratio of consecutive terms in the second series is $-\frac{1}{2}$. This gets at the defining property of a geometric series.

Definition 2.1. A *geometric series* is a series in which the ratio between any two consecutive terms is the same number. This number is called the *common ratio*. In symbols, a geometric series is a series of the form $a + ar + ar^2 + \cdots$. Here, the common ratio is r. This can also be written in the notation $\sum_{n=0}^{\infty} ar^n$.

Regarding this terminology: the reason these series are called *geometric* is due to another description: for a geometric series with positive terms, any given term is the *geometric mean* of the two terms around it, where the geometric mean of x and y is $\sqrt{xy}$. The geometric mean is another notion for the average of two numbers (the usual notion, given by $\frac{x+y}{2}$, is called the *arithmetic mean*). The reason that this is called the geometric mean is due to a certain geometric construction, which is not terribly relevant here.

Here are some more example of geometric series.

$$\begin{aligned}
\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots &\quad (\text{ratio } \tfrac{1}{3}) \\
9+3+1+\frac{1}{3}\cdots &\quad (\text{ratio } \tfrac{1}{3}) \\
1+1+1+1+\cdots &\quad (\text{ratio } 1) \\
1-1+1-1+\cdots &\quad (\text{ratio } -1) \\
2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots &\quad (\text{ratio } \tfrac{1}{4}) \\
1-2+4-8+\cdots &\quad (\text{ratio } -2) \\
9+6+4+\frac{8}{3}+\cdots &\quad (\text{ratio } \tfrac{2}{3})
\end{aligned}$$

Notice, of course, that note all of these series converge. In fact, three of the series above fairly obviously diverge. So we wish to answer two questions:

- When does a geometric series converge?
- If a geometric series converges, what is its sum?

I will begin with the second question.

3 Summing a geometric series (naive first attempt)

In this section, I will be assuming naively that the geometric series I consider converge. In the next section, I will redo this argument more carefully to avoid this dependence.

Recall one geometric series whose sum we do know.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$$

To remind you how this was obtained: imagine this sum as adding up the distance between two points by first taking half the distance, then half of what remains, then half of what remains, and so on. Here is a slightly more precise way to explain this same argument. Notice that the "tail" of the series, omitting the first term, should have sum equal to exactly half of the total sum.

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$

So if the tail is exactly half of the sum, then the other half must be the first term. In symbols, we obtain this by subtraction.

$$\begin{aligned}
\frac{1}{2} &= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)-\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) \\
&= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) \\
&= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)
\end{aligned}$$

Multiplying both sides by 2 gives the result.

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

In fact, the same trick will work with most geometric series. For example, let's evaluate the alternating version of this series in this same way: express the first term of the series as some portion of the entire series. We wish to evaluate $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$.

$$\begin{aligned}
\frac{1}{2} &= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right) - \left(-\frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right) \\
&= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right) \\
&= \left(1 + \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right)
\end{aligned}$$

Now dividing both sides by $1 + \frac{1}{2}$, we get the sum of this series.

$$\begin{aligned}
\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots &= \frac{\frac{1}{2}}{1 + \frac{1}{2}} \\
&= \frac{1}{3}
\end{aligned}$$

I will do one more example before showing the technique in general. Consider the following series.

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

We can do the same trick as before.

$$\begin{aligned}
1 &= \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right) - \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right) \\
&= \left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right) \\
\frac{1}{1 - \frac{1}{3}} &= 1 + \frac{1}{3} + \frac{1}{9} + \cdots
\end{aligned}$$

Therefore, evaluating this gives the following.

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots = \frac{3}{2}$$

It certainly seems like this technique will work in general. Consider an arbitrary geometric series $a + ar + ar^2 + \cdots$. Again, we an determine what part of the whole the term a makes up.

$$\begin{aligned}
a &= \left(a + ar + ar^2 + \cdots\right) - r\left(a + ar + ar^2 + \cdots\right) \\
&= (1 - r)\left(a + ar + ar^2 + \cdots\right) \\
\frac{a}{1 - r} &= a + ar + ar^2 + \cdots
\end{aligned}$$

In fact, this formula will be correct whenever the series converges. However, we certainly still need to decide when the series converges and when it does not. Otherwise, we may make claims like the following.

$$\begin{aligned} 1 - 1 + 1 - 1 + \cdots &= \frac{1}{1-(-1)} \\ &= \frac{1}{2} \end{aligned}$$

In fact, this equation isn't so ridiculous; the partial sums of this series alternative between 1 and 0, so the average partial sum is $\frac{1}{2}$. But in our present situation, the fact is that the series on the left does not converge in the sense that we use in this course.

4 Finite geometric sums

The argument above can be rescued by simply considering finite sums and then taking a limit. This will have the pleasant side effect that convergence will also be able to be determined fairly easily.

Consider the example of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Let's evaluate a partial sum of this series. Consider the sum $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$. We can do the same sort of trick as before: subtract $\frac{1}{2}$ times this sum from the sum.

$$\begin{aligned} \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) &= \frac{1}{2} - \frac{1}{2^{n+1}} \\ \left(1 - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) &= \frac{1}{2} - \frac{1}{2^{n+1}} \\ \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} &= \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} \\ &= 1 - \frac{1}{2^n} \end{aligned}$$

From this we see that the sum of the first n terms of this series covers most of the ground to 1; it is only missing $\frac{1}{2^n}$. Clearly, as n goes to infinity, this goes to 1. So this series converges.

This same computation can be done in general for a partial sum of a geometric series $a + ar + ar^2 + \cdots$.

$$\begin{aligned} \left(a + ar + ar^2 + \cdots + ar^n\right) - r\left(a + ar + ar^2 + \cdots + ar^n\right) &= a - ar^{n+1} \\ (1-r)\left(a + ar + ar^2 + \cdots + ar^n\right) &= a\left(1 - r^{n+1}\right) \\ a + ar + ar^2 + \cdots + ar^n &= a \cdot \frac{1 - r^{n+1}}{1-r} \end{aligned}$$

This formula is important enough that I will put it in a box.

$$\boxed{a + ar + ar^2 + \cdots + ar^n = a \cdot \frac{1 - r^{n+1}}{1-r}}$$

5 Convergence and sums of geometric series

The formula above tells us everything we need to know. In particular, consider what happens as n goes to ∞. If the absolute value of r is less than 1, then the term r^{n+1} will shrink to 0 as n goes to infinity.

$$\lim_{n\to\infty} a \cdot \frac{1 - r^{n+1}}{1-r} = a \cdot \frac{1}{1-r}$$

On the other hand, if r has absolute value of at least 1, then this limit will not exist. Recalling that by definition

$$a + ar + ar^2 + \cdots = \lim_{n\to\infty} \left(a + ar + ar^2 + \cdots + ar^n\right)$$

this now gives the main theorem for geometric series.

Theorem 5.1. *Consider the geometric series* $\sum_{k=0}^{\infty} ar^n$*, with initial term* a *and common ratio* r*. Then:*

- *If* $|r| \geq 1$*, then the series diverges.*
- *If* $|r| < 1$*, then the series converges to* $\dfrac{a}{1-r}$*.*

This is in line with our various examples.

6 The Taylor series of $\frac{1}{1-x}$

Using different symbols, part of the theorem above could also be written as follows. For $|x| < 1$:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{1}$$

This is intriguing because we can consider both sides as functions of x. In fact, we have seen this exact equation (or something suggesting it) before. Indeed, the Taylor series, centered at $x = 0$, for the function $f(x) = \frac{1}{1-x}$ is precisely $1 + x + x^2 + \cdots$. So now we've learned something: *this Taylor series converges if and only if* $|x| < 1$*, in which case it converges to the function* $\frac{1}{1-x}$*.*

I suggest that if you remember any formula from this lecture (or from this entire unit, really), it should be equation 1. Although by remember I don't mean memorize: it really is useful to run through the argument enough times that it becomes second-nature. In other words, I suggest that you really try to *understand* formula 1. Everything else follows from it fairly easily.

With this in hand, various other questions can be answered.

Example 6.1. Consider the series $\sum_{n=1}^{\infty} \dfrac{(x-2)^n}{2^n}$, regarded as a function of x. For which values of x does this series converge? When it converges, what function does it converge to?

The ratio of any two consecutive terms of this series is $\frac{x-2}{2}$. So this is a geometric series, and it converges if and only if $\left|\frac{x-2}{2}\right| < 1$. In fact, this is equivalent to $|x-2| < 2$, which is turn is equivalent to $0 < x < 4$. What is the sum, in this case? The first term of the series is $a = \frac{x-2}{2}$, and the common ratio is $r = \frac{x-2}{2}$, so the sum is $\dfrac{a}{1-r} = \dfrac{\frac{x-2}{2}}{1-\frac{x-2}{2}}$. This simplifies to $\dfrac{x-2}{4-x}$. In fact, this series is the Taylor series of the function $f(x) = \frac{x-2}{4-x}$ centered at $x = 2$.