Lecture 17: The N^{th} term test and the comparison test

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1 Introduction

We are in the process of studying methods for concluding when a series (that is, an infinite summation) converges (that is, is equal to a well-defined finite value). This lecture presents some basic criteria for convergence. In subsequent lectures, we will develop several more criteria, and also consider several key examples (geometric series, p-series, and power series). The main objective is to understand Taylor series.

At the moment, we really only have two example to work with: the harmonic series (and its alternating version $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$), and the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ (and its alternating version). So we will mainly compare to these for now. Soon we will have more examples to work with.

The reading for today consists of three items: Gottlieb §30.5 pp. 969-972 (on the comparison test[1]), the handout "Nth term test, comparison test" (under "reading for the course"), and the solutions to Janet's handout with the same title. The homework is problem set 16 (which includes weekly problem 15). You should begin working on weekly problems 16 and 17.

2 The N^{th} term test

Recall that informally, a series has a well-defined "sum" if there is some number, call it L, such that the partial sums of the series give better and better approximations of L, eventually staying within any chosen margin of error of L.

Thinking in these terms, observe the following: for if n is large enough that the partial sums s_n remain within some margin of error of L, then the terms of the series must be small enough that they cannot possibly push the partial sum out of this margin. In other words, the terms must remain in some margin of error of 0. This suggests the following criterion.

Theorem 2.1 (N^{th} term test)**.** *If the series $a_1 + a_2 + a_3 + \cdots$ converges, then $\lim_{n\to\infty} a_n = 0$.*

The contrapositive of this statement is important enough that I will state it separately.

Corollary 2.2. *If $\lim_{n\to\infty} \neq 0$ (or does not exist at all), then the series $a_1 + a_2 + a_3 + \cdots$ diverges.*

Example 2.3. Consider the series $1 + (-1) + 1 + (-1) + \cdots$. The limit $\lim_{n\to\infty(-1)^n}$ does not exist, so the N^{th} term test implies that this series diverges.

It is very important to note that the converse of theorem 2.1 is not true in general. The classic example of this is the harmonic series.

Example 2.4. Consider the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. This series passes the N^{th} term test since $\lim_{n\to\infty} = 0$, but it still diverges. Thus while failing the test guarantees that a series diverges, passing it does not guarantee that it converges.

[1]This portion of the text unfortunately sometimes invokes various convergence tests that we have not covered yet, due to the different order of the text. You may disregard these remarks for now if you wish.

3 The direct comparison test

We begin with the most basic form of the comparison test. This test will be refined somewhat in section 5. Another version, called the *asymptotic comparison test*, will be presented in a later lecture.

Just like with improper integrals, convergence of series can sometimes be established by bounding a series by a series known to converge (and, by contrapositive, divergence of a series can be established by bounding it below by a series known to diverge).

Theorem 3.1 (Direct comparison test). *Suppose that* $\sum_{k=1}^{\infty} a_n$ *and* $\sum_{k=1}^{\infty} b_n$ *are two series with nonnegative terms such that* $0 \leq a_n \leq b_n$ *for all* n. *Then:*

- *If* $\sum_{k=1}^{\infty} b_n$ *converges, then* $\sum_{k=1}^{\infty} a_n$ *converges.*
- *If* $\sum_{k=1}^{\infty} a_n$ *diverges, then* $\sum_{k=1}^{\infty} b_n$ *diverges.*

Note that I have written these series as starting from $k = 1$, but this is not essential; this test will work no matter what indexing convention is used.

Example 3.2. Consider the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. This can be compared to the harmonic series: for all k, $\frac{1}{\sqrt{k}} \geq \frac{1}{k}$. Since the harmonic series diverges, this series must diverge as well.

Example 3.3. Consider the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$. Then since $\frac{1}{3^n} \leq \frac{1}{2^n}$, we can compare to the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Since the latter series converges, this series converges as well.

4 Initial segments do not matter

An important observation about series is that their convergence (or divergence) does not depend on any finite number of terms: it only depends on the *eventual* behavior of the series. This is a bit of a tricky notion to comprehend: no individual term of the series determines whether it converges, yet all terms taken together do determine whether it converges. This can be expressed by the following.

Theorem 4.1. *Consider any series* $\sum_{n=1}^{\infty} a_n$. *Then for any positive value* N, *this series converges if and only if the "tail"* $\sum_{n=N}^{\infty}$ *converges.*

Therefore to study convergence of a series, is suffices to study convergence of the series after any finite initial segment has been removed.

Example 4.2. Consider the series $\displaystyle\sum_{n=1000}^{\infty} \frac{1}{n}$. This is the same as the harmonic series, except that an initial segment has been removed. Therefore, since the harmonic series diverges, this series diverges as well.

5 The limit comparison test

Bearing in mind that initial segments do not matter, we can refine the direct comparison test to give a test that only uses long-term behavior. The result is the following test.

Theorem 5.1 (Limit comparison test). *Consider two series* $\displaystyle\sum_{k=1}^{\infty} a_k$ *and* $\displaystyle\sum_{k=1}^{\infty} b_k$, *and suppose that for* k *sufficiently large, both series have only positive terms. Suppose also that* $\lim_{k\to\infty} \frac{a_n}{b_n} = L$. *Then:*

- *If* $0 < L < \infty$, *then either both series converge or both series diverge.*
- *If* $L = 0$ *and* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges as well.*
- *If this limit diverges to* $+\infty$ *(or, you could say,* $L = +\infty$*), and* $\sum_{k=1}^{\infty} b_k$ *diverges, then* $\sum_{k=1}^{\infty} a_k$ *diverges as well.*

Here is a rough sketch of the proof of this result. If L is a positive constant, then this means in particular that for sufficiently large k, $a_k/b_k \leq 2L$, so $a_k \leq 2Lb_k$, and also $a_k/b_k \geq L/2$, so that $b_k \leq 2La_k$. Therefore the comparison test can be used in either direction; the convergence (or divergence) of one series implies the convergence (or divergence) of the other. The 0 and ∞ cases are similar, but only work in one direction, since only one bound (upper or lower) can be obtained.

Example 5.2. Consider the series $\sum_{n=27}^{\infty} \frac{n^2+n+1}{3n^3+31n+7}$. It would be a nightmare to try to set of a comparison directly. However, just by looking at this series, is seems like it should be very comparable to the harmonic series. Indeed, taking the ratio and passing to the limit:

$$\begin{aligned} \frac{(n^2+n+1)/(3n^3+31n+7)}{1/n} &= \frac{n^3+n^2+n}{3n^3+31n+7} \\ \lim_{n\to\infty} \frac{n^3+n^2+n}{3n^2+31n+7} &= \lim_{n\to\infty} \frac{1+1/n+1/n^2}{3+31/n^2+7/n^3} \\ &= \frac{1}{3}. \end{aligned}$$

Therefore since both series have positive terms, the limit comparison test applies: the fate of each series is intertwined with the other. Since the harmonic series diverges, so does the series $\sum_{n=27}^{\infty} \frac{n^2+n+1}{3n^3+31n+7}$.

Example 5.3. Consider the series $\sum_{k=52}^{\infty} \frac{7}{2^k-12}$. This looks like it could be compared with the familiar geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. Indeed, both series are positive, and the ratios of corresponding terms are $\frac{7\cdot 2^k}{2^k-12}$, which goes to 7 as $k \to \infty$.Therefore one converges if and only if the other does; since we know that $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ converges, then also $\sum_{k=52}^{\infty} \frac{7}{2^k-12}$ converges.