Lecture 16: Convergence of series

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1 Introduction

Recall that a "series" is simply an infinite sum. Our most important examples will be Taylor series. In order to study Taylor series more carefully, it is necessary to examine series in general more carefully. We will begin this today; it will occupy us into November.

As is always the case when dealing with the infinite, some care is needed. In particular, an infinite sum can only be said to have a well-defined value in special circumstances, when it is said to converge. This lecture defines convergence and states some principal examples and basic techniques for studying them, including the notions of absolute and conditional convergence. In subsequent lectures, more techniques will be introduced, and more examples described.

We will be jumping around in the text for the next several days, covering all the topics in a different order.

The reading for today is the following four items: the first part of Gottlieb §30.5, on monotone series (the first two pages only, up to "the integral test"), the first part of Gottlieb §30.4, on absolute and conditional convergence (up to "alternating series"), the handout "Definition of Convergence" (under "reading for the course"), and the solutions to Janet's worksheet "Definition of Convergence" (under "additional resources"→ "worksheets." The homework is problem set 15 (which includes weekly problem 14) and a topic outline.

2 Prologue

When studying infinite series, we are trying to make sense of the ways in which the accumulation of infinitely many things can lead to a finite whole. For example, what are we to make of claims like the following?

$$\begin{aligned} e &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \end{aligned}$$

In fact, we have already seen a concept that is very much analogous to this: improper integrals. An integral over an infinite domain can be defined as a limit of integrals over finite domains.

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

The same idea will work for series.

Recall some of the subtleties that arose in this context, as exemplified by the following examples.

$$\int_1^\infty \frac{1}{x^2} dx \quad \text{Converges to a finite value (1).}$$

$$\int_1^\infty \frac{1}{x} dx \quad \text{Diverges to infinity, despite a very similar shape to the above.}$$

$$\int_0^\infty \sin x dx \quad \text{Diverges due to oscillation.}$$

The same sort of phenomena will also occur in the context of series. In fact, there are cases where series can be studied using improper integrals, and vice versa; we will see one example, in the form of the "integral test," later in the course.

3 Definition of convergence

Just like with improper integrals, the definition of the value of a series proceeds from a known notion – finite sums – and proceeds to series by taking a limit. The series is said to converge or diverge according to whether this limit exists (and is finite) or not.

Definition 3.1. The sum of an infinite series is

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

If this limit exists, the series is said to *converge*. If the limit does not exist, the series is said to *diverge*.

It is worth pointing out that some authors refer to a series where this limit goes to ∞ or $-\infty$ as converging to ∞ or $-\infty$. There is a way to may this precise. It is a matter of terminology whether such series are said to converge or not. For our purposes, we will say that they diverge (or sometimes, "diverge to ∞" or "diverge to $-\infty$).

We will often speak about the partial sums of a series.

Definition 3.2. The *nth partial sum* of a series is the sum of the first n terms in the series. It is often denoted by s_n.

Notice that the partial sums of a series form a sequence, and the limit of this sequence (as n goes to infinity) is the sum of the series.

4 Zeno's paradox

Thinking about infinite series leads to various paradoxes that have been historically notable. I will mention one form of Zeno's paradox, which traditionally goes back Zeno of Alea, a pre-Socratic philosopher. This paradox played a significant role in the work of Aristotle, and continues to be an important example.

Another form of Zeno's paradox is described in the excerpt from Tolstoy that I uploaded earlier, concerning Achilles and the tortoise. We consider the form that is stated in terms of a traveling arrow.

The paradox can be stated as follows. Suppose that an arrow is traveling the distance between an archer and a target. Then the arrow must do all of the following things:

- It must travel half of the distance.
- It must travel half of the remaining distance ($\frac{1}{4}$ of the total).
- It must travel half of the remaining distance ($\frac{1}{8}$ of the total).
- ...

The trouble is that this list is infinite in principle. Therefore, to travel all the way to the target, the arrow must travel an infinite number of steps in a finite amount of time. This, Zeno concludes, is impossible, and therefore all movement is an illusion.

What does this have to do with series? We can think about the distance that the arrow travels after it completes some number of these steps. After 1 step, is has traveled $\frac{1}{2}$ of the way. After 2 steps, it has traveled $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ of the way. After 3 steps, it has traveled $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ of the way. A pattern is starting to emerge. After n steps, the total portion of the distance to the target that the arrow has traveled is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

From the first couple guesses, you might conclude that this should always be equal to $\frac{2^n-1}{2^n}$. This is correct. This can also be written more suggestively as $1 - \frac{1}{2^n}$.

How you resolve this paradox from a philosophical standpoint is up to you. For me, the resolution is this: *it is possible to do an infinite number of things in a finite amount of time.* This was a surprisingly difficult idea to swallow, historically. In any case, here is a question: after performing the infinitude of steps described above, how far will the arrow have travelled?

The answer should be the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, or alternatively $\sum_{k=1}^{\infty} \frac{1}{2^n}$. By our definition in the previous section, this should be $\lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right)$, which is 1. So upon carrying out these infinitely many steps, the arrow has travelled the entire distance to the target. This makes sense: if any given point between the archer and target is chosen, the arrow eventually reaches that point. So it covers the entire distance between the two points (except, in some sense, the exact point of hitting the target, but this has "length 0").

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

This is the most basic example of a geometric series. All of the ideas of geometric series in general are contained within it; we will discuss the more general case in a later class.

As a slightly more complicated example, consider this infinite series.

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots$$

Will this also converge to something? It is possible to make up a story in this case as well. Suppose that you are standing on a long field, and you want to walk one third of the way across it, but you are only capable of measuring half of known distances. Maybe you overshoot, by walking halfway across it. Now you need to walk back one third of the way across the distance you just traveled. Again, you overshoot by walking back over half of the distance just traveled ($\frac{1}{4}$ of the total field). Now you've overshot again, and need to travel back one third of the way back over that distance; so walk half of the way back over that distance ($\frac{1}{8}$ of the total field). Continuing in this way, the distance from the end that you started at will be $\frac{1}{2}$, then $\frac{1}{2} - \frac{1}{4}$, then $\frac{1}{2} - \frac{1}{4} + \frac{1}{8}$, and so on. But the distance to $\frac{1}{3}$ is getting smaller and smaller. Thus after performing all of these steps, you will converge to $\frac{1}{3}$. This rough sketch suggests that

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots = \frac{1}{3}$$

We will later study geometric series, and how to sum them, more systematically.

5 What is $0.\bar{9}$?

It is amazing how much people are capable of arguing (especially on the internet) about the following question: what number is $0.\bar{9}$? If this notation is not familiar, it denotes $0.9999999\ldots$, where the 9s go on forever.

Expressed in terms of a series, this question can be rephrased: what is the value of the following infinite sum?

$$0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

Using Σ notation, the question becomes: what is $\sum_{k=1}^{\infty} \frac{9}{10^k}$?

There is only one possible number that makes sense: 1. But the usual objection is: "how can the sum be 1, if the terms never actually reach it?" This question really gets to the heart of what series and convergence are all about.

What is true is that the nth partial sum, that is, the sum of the first n terms, is $1 - \frac{1}{10^n}$. Of course, this is less than 1. As n goes to infinity, this value becomes arbitrarily close to 1 itself. The limit, then, is $\lim_{n\to\infty}(1 - \frac{1}{10^n}) = 1$. Therefore $\sum_{k=1}^{\infty} \frac{9}{10^k} = 1$, and $0.\bar{9} = 1$ also.

The point here is that the sum of a series is not any particular partial sum: it is what lies at the end of all of the partial sums; it is the value that the partial sums become arbitrarily close to, whether they ever reach it or not. This idea is expressed by the concept of a limit.

Regarding the controversial decimal $0.9999\ldots$ itself: it is true that whether this is equal to 1 is in some sense a matter of notation. However, if you wish to say that this decimal is not equal to the number 1, you have a heavy burden of explaining what, if not a number, this thing is meant to represent. The important distinction, which is very easy to forget, is that *there is a difference between a number in itself and a decimal representation of that number*. A number is just a quantity. It is a quantity whether we write it in any symbols or not. Decimals are symbols used to describe numbers, but they cannot do it perfectly; all they can do is provide a series of better and better approximations (for example, pi is not 3, nor 3.1, nor 3.14, but it *is* the limit of an infinite sequence that begins with these values and adds one more decimal place each time).

I also will remind you that this discussion of decimals is very much analogous to how you should view Taylor series. Each Taylor approximation gets a little bit closer to a function. None ever reaches it, but the Taylor *series*, in its totality and regarded as a limit, may well get all the way there, just as the infinite decimal $0.9999\ldots$ does get all the way to 1 as a limit.

As an aside: one standard response to the argument above is that $0.999\ldots$ is infinitely close to 1, but not quite 1. You might say that it is $1 - \epsilon$, where ϵ is infinitely small, but not 0. Quantities like this ϵ are called *infinitesimals*, and it is possible to make them coherent. However, it requires some care to do this properly, since many apparent paradoxes will arise. The study of infinitesimal numbers forms an important part of the subject of *nonstandard analysis*, which many people prefer as an alternative to the usual foundations of calculus.

6 The harmonic series

The following series is traditionally called the harmonic series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

In Σ notation, this is written $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

The term "harmonic" comes from the fact that when a string of length 1 is plucked, it produces sounds at wavelengths $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$ (though by no means all with equal energy). The properties of the series are not really very related to the properties of such strings.

Does this series converge or diverge? It looks like in might converge, since the terms go to 0. On the other hand, the function $f(x) = \frac{1}{x}$ goes to 0 as x goes to ∞ as well, but its improper integral diverges.

In fact, we will later cover the integral test, which shows that the divergence of $\int_1^{\infty} \frac{1}{x} dx$ implies the divergence of the harmonic series as well. In fact, the nth partial sum of the harmonic series is very close to $\ln n$; in particular, while the partial sums do diverge to infinity, they do it quite slowly.

Here is an elementary argument for why the harmonic series diverges.

- There are 2 terms that are at least $\frac{1}{2}$ (1 and $\frac{1}{2}$).
- There are 2 terms that are smaller than $\frac{1}{2}$, but at least as large as $\frac{1}{4}$ ($\frac{1}{3}$ and $\frac{1}{4}$). Their sum must be at least $2 \cdot \frac{1}{4} = \frac{1}{2}$.
- There are 4 terms that are smallest than $\frac{1}{4}$, but at least as large as $\frac{1}{8}$ ($\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, and $\frac{1}{8}$). Their sum is at least $4 \cdot \frac{1}{8} = \frac{1}{2}$.
- There are 8 terms that are smaller than $\frac{1}{8}$, but at last as large as $\frac{1}{16}$. Their sum is at least $8 \cdot \frac{1}{16} = \frac{1}{2}$.
- ...

Continuing this process, we will obtain an infinite number of "blocks" of the series, each of which has sum at least $\frac{1}{2}$. Therefore, the total sum is at least $\frac{1}{2}$, and at least $\frac{1}{2} + \frac{1}{2}$, and at least $\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$, and so on. So in fact, this sum diverges to infinity.

Now consider the harmonic series's more tolerable sister, which is called the *alternating harmonic series.*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

In Σ notation, this is $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$.

This series, in fact, converges. Here is how to see this: the partial sums move to the left, then the right, then the left again, and so on. Adding each term retraces part, but not all, of the distance added by the previous term. So the partial sums are zeroing in on something. Since the terms themselves go to 0, this zeroing in gets more and more precise, and must converge to some limit. This argument will be made precise later by the alternating series test.

In fact, the alternating harmonic series should be familiar: it is the Taylor series for $\ln x$, with center $c = 1$, evaluated at $x = 2$. So the sum of the alternating harmonic series is, in fact, the natural logarithm of 2.

7 Monotone series

A series is called *monotone* if all of its terms are positive, or all of its terms are negative.

The nice thing about monotone series is that they cannot oscillate at all; if they diverge, then they diverge to $\pm\infty$. This is expressed by the following principle.

Theorem 7.1. *A monotone series converges if and only if its partial sums are bounded.*

Here "bounded" means that there is some finite interval in which all of the partial sums lie. We mention two examples for now.

First, consider the harmonic series. It is monotone (all terms are positive), and its partial sums are unbounded (they grow arbitrarily large). Therefore, by this theorem, the harmonic series diverges.

Second, consider the series $\sum_{k=1} \infty (-1)^k$. This can be written $-1 + 1 - 1 + 1 - 1 + \cdots$. The partial sums are all either -1 or 0, so they are bounded. But the series is *not* monotone, since the terms change between positive and negative sign. Therefore, this theorem does not apply. In fact, this series diverges.

8 Absolute convergence

There is a fundamental difference between the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$. Both have the same terms, but different signs. As a result, one diverges, while the other converges. In some sense, the alternating series gets lucky: although it has no right to converge based on the sizes of its terms, it still gets away with converging by cleverly interlacing positive and negative signs.

It happens that it is often useful to know whether a series converges by this sort of "luck" or not. Thus we define another notion, which is a more strict condition than convergence.

Definition 8.1. A series $\sum_{k=1}^{\infty} a_n$ is said to *converge absolutely* if the sum of the absolute values $\sum_{k=1}^{\infty} |a_n|$ converges.

It happens that a series which converges absolutely is guaranteed to converge. However the converse is not true: the alternating harmonic series converges, but it does not converge absolutely (since the harmonic series diverges). There is a word for this as well.

Definition 8.2. A series with converges, but which does not converge absolutely, is called *conditionally convergent.*

These three notions are summarized in this table. Notice that if a series has all positive terms, then convergence and absolute convergence mean the same thing.

Term	Definition	Example
Absolutely convergent	$\sum_{k=1}^{\infty} \lvert a_n \rvert$ converges (and therefore so does $\sum_{k=1}^{\infty} a_n$).	$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$
Conditionally convergent	$\sum_{k=1}^{\infty} a_n$ converges, but $\sum_{k=1}^{\infty} \lvert a_n \rvert$ does not.	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
Divergent	$\sum_{k=1}^{\infty} a_n$ diverges (and therefore so does $\sum_{k=1}^{\infty} \lvert a_n \rvert$).	$1 - 1 + 1 - 1 + \cdots$

So as not to leave this notion entirely unexplained: the basic reason that conditionally convergent series must sometimes be handled with care is a general principle: *the value of a series that is only conditionally convergent depends not just on the terms of the series, but on the order that they appear. The value of an absolutely convergent series does not depend on the order in which the terms appear.* You are not responsible for knowing this principle, so I will not go into it in detail. But I do wish to mention is as an explanation of why this notion turns out to be important in mathematics.