# Math 1B, lecture 13: Taylor approximation

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5 October 2011

*I am amazed that it occurred to no one (if you except N. Mercator with his quadrature of the hyperbola) to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences... And just as the advantage of decimals consists in this, that when all fractions and roots have been reduced to them they take on in a certain measure the nature of integers; so it is the advantage of infinite variable-sequences that classes of more complicated terms... may be reduced to the class of simple ones: that is, to infinite series of fractions have simple numerators and denominators and without the all but insuperable encumbrances which beset the others.*

Sir Isaac Newton, *De Methodis*[1]

## 1 Introduction

As the quotation above suggests, Taylor approximation, in some sense, is to functions what decimal approximation is to numbers. For basic purposes. 3 is a pretty good approximation of the otherwise complicated number $\pi$, while 3.1 is a little better, and 3.14 is better still. The point is that each new approximation gives a little bit more accuracy, but remains very simple and easy to compute with. In similar fashion, we will see that $1+x$ is a pretty good approximation of the function $e^x$, while $1+x+\frac{1}{2}x^2$ is better, and $1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$ is better still. These approximations have the benefit that they are easy to compute (by a computer, or even by hand), and also easy to manipulate algebraically. The subject of the next several lectures will be to describe how to obtain approximations like these, and how well they approximate more complicated functions. This will lead into a more general discussion of sequence and series, which will constitute the middle of the three main parts of the course.

There are two main reasons to study Taylor approximation: one is quantitative, and one is qualitative. The quantitative reason is that Taylor approximations are very practical ways to compute functions like $e^x$ or $\cos x$ by hand or on a machine. If you have ever wondered how it is possible to calculate decimal places of numbers like $e$ or $\pi$, you will see ways very soon. The qualitative reason is that the studying Taylor approximations sheds light on two fundamental questions:

- How much of a function's behavior is encoded only at its derivatives at a single point?
- How well can a function's behavior be modeled by a polynomial?

In practice, you are unlikely to actually write down a Taylor approximation by hand. Far more interesting, in my opinion, is the qualitative aspect: it reduces many questions in mathematics to considering the properties of only the most fundamental functions, namely polynomials.

The most important point of this lecture, which will be elucidated once definitions are made, is that a Taylor approximation simply records in a polynomial the first several derivatives of a function at a point.

*The data of a degree $n$ Taylor approximation are equivalent to the values of the function and its first $n$ derivatives at a single point.*

[1]Despite their name, Taylor series were really pioneered in the work of Newton. The primary contribution of Brook Taylor was to give an error bound, known as Taylor's theorem, which we will discuss next time.

It is worth noting that a degree 1 Taylor approximation is the same thing as a linear approximation. We will therefore begin by discussing linear approximation.

The reading for today is Gottlieb §30.1, as well as the "Taylor Approximation" handout (located under "Reading for the course" on the webpage). The homework is problem set 12 and a topic outline. You should begin working on weekly problem number 14.

## 2 Dramatic foreshadowing

Consider the first bulleted question in the introduction. One of the main lessons of Taylor series is that, for well-behaved functions, the values of the derivatives at a single point tell a great deal about the total behavior of the function. Consider three of the hallmark functions of calculus: $e^x, \sin x$, and $\cos x$. Their derivatives are easy to compute to the end of time.

$$\begin{array}{rrr}
f(x) = e^x & g(x) = \sin x & h(x) = \cos x \\
f'(x) = e^x & g'(x) = \cos x & h'(x) = -\sin x \\
f''(x) = e^x & g''(x) = -\sin x & h''(x) = -\cos x \\
f'''(x) = e^x & g'''(x) = -\cos x & h'''(x) = \sin x \\
f^{(4)}(x) = e^x & g^{(4)}(x) = \sin x & h^{(4)}(x) = \cos x \\
f^{(5)}(x) = e^x & g^{(5)}(x) = \cos x & h^{(5)}(x) = -\sin x \\
\vdots & \vdots & \vdots
\end{array}$$

Are you bored yet? The point is that for all three functions, the derivatives eventually start repeating themselves. As an aside (to which we will return in about a month), observe that both the functions $\sin x$ and $\cos x$ have the property that their second derivative is the negative of themselves. We will see that this property is, in some sense, *characteristic* of these functions, in the same sense that having some higher derivative identically equal to 0 is characteristic of polynomials.

Now, focus attention entirely on $x = 0$. Here is what we see.

$$\begin{array}{rrr}
f(x) = e^x & g(x) = \sin x & h(x) = \cos x \\
f(0) = 1 & g(0) = 0 & h(0) = 1 \\
f'(0) = 1 & g'(0) = 1 & h'(0) = 0 \\
f''(0) = 1 & g''(0) = 0 & h''(0) = -1 \\
f'''(0) = 1 & g'''(0) = -1 & h'''(0) = 0 \\
f^{(4)}(0) = 1 & g^{(4)}(0) = 0 & h^{(4)}(0) = 1 \\
f^{(5)}(0) = 1 & g^{(5)}(0) = 1 & h^{(5)}(0) = 0 \\
f^{(6)}(0) = 1 & g^{(6)}(0) = 0 & h^{(6)}(0) = -1 \\
f^{(7)}(0) = 1 & g^{(7)}(0) = -1 & h^{(7)}(0) = 0 \\
\vdots & \vdots & \vdots
\end{array}$$

There is a nice pattern emerging. The sequence of derivatives at 0 of $e^x$ is very simple: they are all 1. The derivatives at 0 for $\sin x$ and $\cos x$ are a little more interesting, but still fairly simple: they alternative between being 0 and $\pm 1$, with the sign alternating.

These sequences of numbers can be regarded as a *signature* of sorts for these three functions. A question which will concern us going forward is: suppose I only give you one of these signatures (such as the signature $0, 1, 0, -1, 0, 1, 0, -1, \dots$). Is this enough to tell the function that I have in mind? The answer, remarkably, is essentially yes.

*Observation* 2.1. It absolutely critical in the above that $\sin x$ and $\cos x$ are defined using *radians*, rather than *degrees*. In fact, to my mind, these simple signatures are the primary reason to use radians in the first place. Similarly, it is critical that we consider $e^x$, not $10^x$ or some other exponential, since otherwise the signature would be much less symbol. This is behind the supremacy of $e$ as the base of exponential functions.

It will soon be clear why I have computed these sequences.

## 3 Linear approximation

Recall one definition of linear approximation of a function $f$. We will use the notation $P_1$ in order to match notation used in the following section.

**Definition 3.1.** The linear approximation $P_1(x)$ of a function $f(x)$ at a point $c$ is the unique linear function $P_1(x)$ such that $P_1(c) = f(c)$ and $P_1'(c) = f'(c)$.

In symbols, the linear approximation can be written as follows.

$$P_1(x) = f(c) + (x - c)f'(c)$$

Notice that all this is is a linear function that is specifically jury rigged to match $f(x)$ and $f'(x)$ at the value $x = c$.

One may to consider linear approximation that I find particularly compelling is to consider the original definition of the derivative at $c$.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The definition of the limit means nothing but the fact that, for $x$ very close to $c$:

$$f'(c) \approx \frac{f(x) - f(c)}{x - c}$$

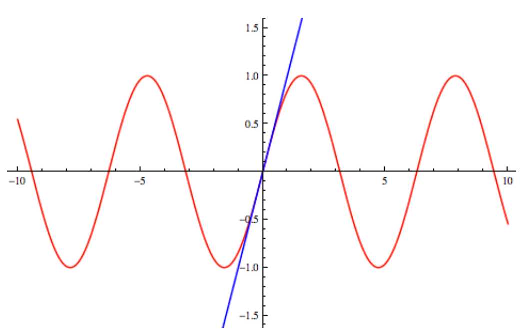
and rearranging this, it follows that for $x$ very close to $c$:

$$f(c) + (x - c)f'(c) \approx f(x).$$

Therefore, we have the following perspective on linear approximation: *the faster the limit defining the derivative of $f$ at $c$ converges, the faster the linear approximation becomes a good approximation of the function.*

Graphically speaking, the linear approximation approximates the the graph of a function by the line tangent to the graph of the function.

*Example* 3.2. The linear approximation to $f(x) = \sin x$ at $x = 0$ is $R_1(x) = x$.



# 4 Taylor Approximation

Taylor approximation is a generalization of linear approximation. It is defined in essentially the same way.

**Definition 4.1.** The $n^{th}$ order Taylor approximation $P_n(x)$ of a function $f(x)$ at a point $c$ is the unique degree $n$ polynomial $P_n(x)$ such that $P_n(c) = f(c), P_n'(c) = f'(c), \ldots, P_n^{(n)}(c) = f^{(n)}(c)$. In words, it is the unique degree $n$ polynomial which matches the value and first $n$ derivatives of $f$ at $x = c$.

*Observation* 4.2. Giving the polynomial $P_n(x)$ is equivalent to giving the value and first $n$ derivatives of $f(x)$ at $x = c$.

It is necessary to express this definition in a more "computable" form. Doing this rests on the observation that the function $f(x) = x^n$ has the property that $f^{(n)}(x) = n(n-1)\cdots 3 \cdot 2 \cdot 1$, which is usually denoted $n!$. So in particular $f^{(n)}(0) = n!$. But also note that $f(0) = 0$ and that all other derivatives of $f$ at $x = 0$ are 0. Similarly, if we translate this function to be centered on the point $x = c$, we obtain the function $f(x) = (x-c)^n$, which has the property that $f^{(n)}(c) = n!$, but $f(c) = 0$ and all other derivatives of $f$ are 0 at $x = c$.

Taking advantage of this observation to construct the polynomialals $P_n(x)$ "term by term," the following formula is obtained for the Taylor approximation centered at $x = c$.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

This is often written using $\Sigma$ notation as follows.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

# 5 Examples

Here are some examples of Taylor approximations.

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*Example* 5.1. $f(x) = e^x$, centered at $x = 0$. As discussed in section 2, all derivatives of $f(x)$ at $x = 0$ are 1. So the Taylor approximations at $x = 0$ have a particularly simple form.

$$\begin{aligned} P_n(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \\ &= \sum_{k=0}^{n} \frac{1}{k!}x^k \end{aligned}$$

In many ways, this is the prototypical Taylor approximation; all of the key aspects of the structure of Taylor approximations can be found here.

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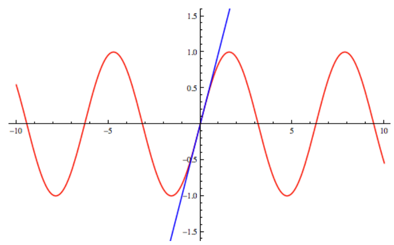
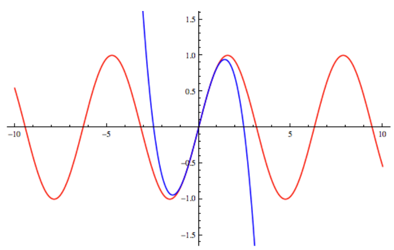
*Example* 5.2. $f(x) = \sin x$, centered at $x = 0$. Since every other derivative of $f(x)$ vanishes at $x = 0$, the Taylor approximation only contains every other term. The first several approximations are:

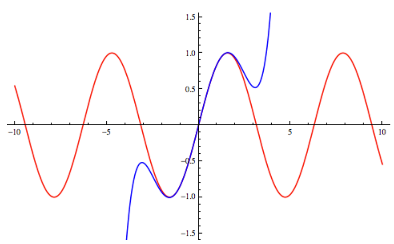
$$\begin{aligned} P_1(x) &= x \\ P_2(x) &= x \\ P_3(x) &= x - \frac{1}{6}x^3 \\ P_4(x) &= x - \frac{1}{6}x^3 \\ P_5(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{aligned}$$

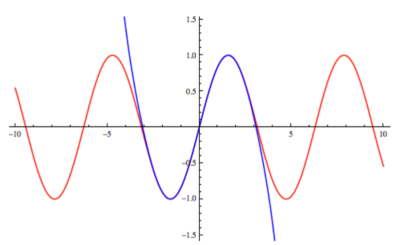
Note that $P_1(x) = P_2(x)$ and $P_3(x) = P_4(x)$ hold because the second and fourth derivatives at $x = 0$ are 0.

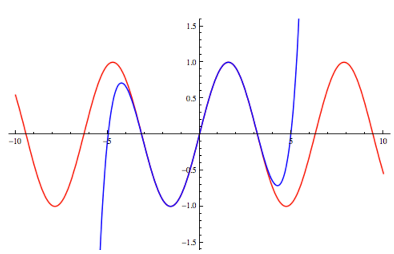
Here are some graphs of successive Taylor approximations of $\sin x$ centered at $x = 0$. It is quite amazing to see how they wrap closer and closer to the function as more terms are included in the approximation.

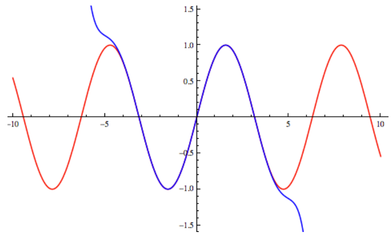





*Example* 5.3. $f(x) = \cos x$, centered at $x = 0$. This is vert similar to the previous example.

$$\begin{aligned}
P_1(x) &= 1 \\
P_2(x) &= 1 - \frac{1}{2}x^2 \\
P_3(x) &= 1 - \frac{1}{2}x^2 \\
P_4(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \\
P_5(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \\
P_6(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6
\end{aligned}$$

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*Example* 5.4. $f(x) = \ln x$.

In this case, centering the approximation at $x = 0$ is not going to work, since the function is not even defined there. But there is a natural choice: center instead at $x = 1$, where $f(x) = 0$. Observe that:

$$\begin{aligned}
f(x) &= \ln x \\
f'(x) &= x^{-1} \\
f''(x) &= -x^{-2} \\
f'''(x) &= 2x^{-3} \\
&\cdots \\
f^{(n)}(x) &= (-1)^{n-1}(n-1)!\ x^{-n}
\end{aligned}$$

From this, it is easy to evaluate all the derivatives at $x = 1$.

$$\begin{aligned}
f(1) &= 0 \\
f'(1) &= 1 \\
f''(1) &= -1 \\
f'''(1) &= 2 \\
&\cdots \\
f^{(n)}(1) &= (-1)^{n-1}(n-1)!
\end{aligned}$$

Now from this, it happens that the Taylor approximation centered at $x = 1$ has a fairly simple form.

$$\begin{aligned}
P_n(x) &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n \\
&= \sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k}(x-1)^k
\end{aligned}$$

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*Example* 5.5. Approximate $\sqrt{5}$ by hand, using a second order Taylor approximation.

Consider the function $f(x) = \sqrt{x}$. We want to know $f(5)$. We happen to know $f(4) = 2$, and it is easy enough to compute derivatives of $f(x)$: $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$. Therefore we have:

$$\begin{aligned} f(4) &= 2 \\ f'(4) &= \frac{1}{4} \\ f''(4) &= \frac{1}{32} \end{aligned}$$

And therefore we have linear and $2^{nd}$ order Taylor approximations around $x = 4$ as follows.

$$\begin{aligned} P_1(x) &= 2 + \frac{1}{4}(x-4) \\ P_2(x) &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \end{aligned}$$

In particular, these give approximations of $\sqrt{5} = f(5)$.

$$\begin{aligned} P_1(5) &= 2 + \frac{1}{4} \\ &= 2.25 \\ P_2(5) &= 2 + \frac{1}{4} - \frac{1}{64} \\ &= 2 + \frac{15}{64} \\ &= 2.234375 \end{aligned}$$

The actual value of $\sqrt{5}$ is approximately $\sqrt{5} \approx 2.23606798$.

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