Math 1B, lecture 12: Improper integrals II

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1 Introduction

As mentioned in the previous lecture, there are two ways in which an integral can be improper. The first is the case that was discussed last time: the domain of integration is infinite. The second case, discussed this time, is when the integrand goes to infinity somewhere in the domain of integration, or when the integrand is otherwise undefined or discontinuous at one or more points.

Again, the objective is the same: to devise an adequate definition that gives a meaningful and well-defined value to these integrals when possible (in which case we say that the integral converges), and to determine when no such value exists (in which case we say that the integral diverges).

The reading for today is the rest of Gottlieb §29.4. The homework is problem set 11 and a topic outline.

2 Definitions

First, we define which integrals, precisely, we consider to be improper.

As usual, the integral should signify the area under the given curve, even when the definition of the curve seems to preclude the usual definition using Riemann sums. In some cases, no such "area" makes sense.

Definition 2.1. An integral $\displaystyle\int_c^d f(x)dx$ is considered *improper* if the function $f(x)$ is discontinuous, infinite, or otherwise undefined at one or more points in the interval $[a, b]$. These points are called *improprieties* of the integral.

As with integral which are improper due to infinite domains of integration, these improper integrals are also defined in terms of limits of ordinary Riemann integrals. First consider an integrals with only one impropriety, at one of the limits of integration.

Definition 2.2. Suppose that $\displaystyle\int_c^d f(x)dx$ has only one impropriety, at the left endpoint $x = c$. Then define

$$\int_c^d f(x)dx = \lim_{b\to c^+} \int_b^d f(x)dx.$$

Similarly, if the integral has only one impropriety at the right endpoint $x = d$. Then define

$$\int_c^d f(x)dx = \lim_{b\to d^-} \int_c^b f(x)dx.$$

In both cases, the integral is said to *converge* if the limit converges to a finite value, and the integral is said to *diverge* otherwise.

In order to evaluate an integral which has multiple improprieties, it is necessary first to split the integral into several pieces, so that each impropriety is resolved by its own limit, and then summing the pieces. The following definition describes the definition of improper integrals in general.

Definition 2.3. An integral with multiple improprieties (and possibly with infinite domain of integration) is defined to be the sum of several improper integrals, each of the same function, but splitting the domain of integration into several pieces so that

- Each piece contains at most one impropriety, which is at one of the limits of integration, and
- Each piece whose interval is infinite has no improprieties and is infinite in at most one direction.

If all of the pieces converge, then the integral is said to *converge* and have value equal to the sum of the value of each piece. If any of the pieces diverges, then the integral is said to *diverge*.

Notice that this definition encompasses the case of integrals of the form $\int_{-\infty}^{\infty} f(x)dx$, which similarly must be split up into two integrals, each evaluated with a limit.

In this definition, it does not matter how this "splitting" is accomplished. It is worth considering why this is the case, but I will not elaborate on it here.

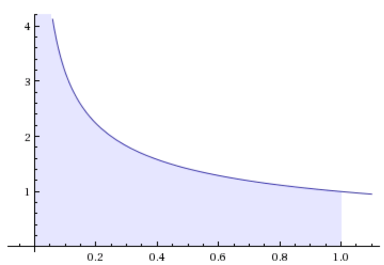
3 Examples

Here are some examples of integrals which are improper due to the integrand going to infinity.

Example 3.1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$

The impropriety is at $x = 0$, where the integrand goes to infinity.

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt{x}}dx \\ &= \lim_{b\to 0^+}\left[2\sqrt{x}\right]_b^1 \\ &= \lim_{b\to 0^+}(2-2\sqrt{b}) \\ &= 2.\end{aligned}$$



Observe that in this case, this area could also be calculated as $1 + \int_1^{\infty} \frac{1}{y^2}dy$, by slicing the area under the curve horizontally rather than vertically.

Example 3.2. $\displaystyle\int_{-1}^1 \frac{1}{x}dx$

The impropriety is at $x = 0$, so this must be split into two improper integrals.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x} dx &= \int_{-1}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx \\
&= \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x} dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x} dx \\
&= \lim_{b\to 0^-} [\ln|x|]_{-1}^{b} + \lim_{b\to 0^+} [\ln|x|]_{b}^{1} \\
&= \lim_{b\to 0^-} \ln|b| - \lim_{b\to 0^+} \ln|b|
\end{aligned}$$

Both of these limits diverge to infinity, so this integral **diverges**.

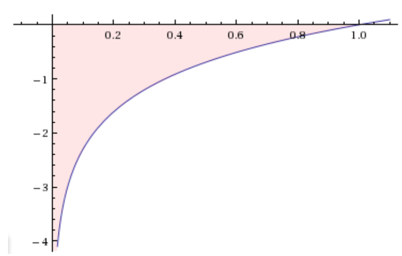
It is a reasonable question why the two halves do not simply "cancel" each other, and leave 0 as the value of the integral. The reason is that the arithmetic this would require is $\infty - \infty = 0$, which would lead to various problems. I will not dwell on this point here, but it is worth thinking about.

Example 3.3. $\displaystyle\int_0^1 \ln x dx$

The impropriety here is at $x = 0$, where the integrand goes to negative infinity.

$$\begin{aligned}
\int_0^1 \ln x dx &= \lim_{b\to 0^+} \int_b^1 \ln x dx \\
&= \lim_{b\to 0^+} [x\ln x - x]_b^1 \\
&= \lim_{b\to 0^+} b\ln b - \lim_{b\to 0^+} (1-b) \\
&= \lim_{b\to 0^+} b\ln b - 1 \\
&= -1.
\end{aligned}$$

In the final step, it is necessary to evaluate the limit $\lim_{b\to 0^+} b\ln b$, which is equal to 0 (for example, using l'Hôpital's rule).



Note that by "slicing horizontally," this area could also be calculated equally well as $-\int_{-\infty}^{0} e^y dy$.

Example 3.4. $\displaystyle\int_0^1 x^p dx$, where p is some constant.

Of course, the answer will depend on p. For $p \geq 0$, there is no impropriety, and the value of the integral is easily calculated to be $\frac{1}{p+1}$.

For $p < 0$, there is an impropriety at $x = 0$. In this case, as long as $p \neq -1$:

$$
\begin{aligned}
\int_0^1 x^p dx &= \lim_{b\to 0^+} \int_b^1 x^p dx \\
&= \lim_{b\to 0^+} \left[\frac{1}{p+1} x^{p+1}\right]_b^1 \\
&= \frac{1}{p+1} - \frac{1}{p+1} \lim_{b\to 0^+} b^{p+1}
\end{aligned}
$$

Now, this last limit is equal to 0 if $p+1>0$, and it diverges if $p+1<0$.

The remaining case is $p=-1$, which we saw in an earlier example gives a divergent integral also. Putting this together: this integral diverges is $p\le -1$, but converges to $\frac{1}{p+1}$ in all other cases.
