Math 1B, lecture 10: Three-dimensional density

Nathan Pflueger

28 September 2011

1 Introduction

This lecture is essentially a combination of earlier discussions of density with later discussions of volumes of three-dimensional solids. The basic problem can be states as follows: given a three-dimensional object containing some substance whose density varies, how can we compute the total mass of the substance? This generalizes the discussion of volume, since the total mass, if the density is 1 everywhere, is simply the volume of the solid.

The main difference between these problems and volume problems is that the density function will generally force a particular slicing scheme (namely, the shape should be sliced into regions of nearly constant density), whereas when computing volumes, any convenient slicing scheme will work.

These ideas are generalized somewhat in multivariable calculus. The main difference between the problems we consider and more sophisticated problems is that the regions will always be able to be sliced using only one slicing variable.

As in all of the other interpretations of integration discussed, you should regard this only as a family of examples of integration. What is important is not to remember the rules and formulas, but to understand how they are constructed and why they lead nicely into a formulation using integrals.

The reading for today is Gottlieb §27.1. The homework is problem set 8 (which includes weekly problems 7 and 8) and a topic outline. You should also begin working on weekly problems 10 and 11.

These notes contain some examples and general methods. In class we considered the problems on Janet Chen's worksheet, which can be found on the course website under "additional resources"→"worksheets."

2 The general technique

Suppose that some substance (say a mineral) is distributed throughout a three-dimensional object. Suppose further that the density of the substance at a point of the object is given by $\rho(x)$, where x is some variable that varies between a and b in the object. Suppose further that the object can be "sliced along x" so that the volume of the slice with x in a very short interval $[x_{k-1}, x_k]$ is approximately $A(x)\Delta x$, where x is any value in this short interval and $\Delta x = x_k - x_{k-1}$ is the width of the interval. Then the total mass of the substance can be approximate by:

$$\text{(Total mass)} \quad \approx \quad \sum_{k=1}^{n} A(x_k)\rho(x_k)\Delta x,$$

where as usual:

$$\begin{aligned} x_0 &= a \\ x_n &= b \\ \Delta x &= (b-a)/n \\ x_k &= a + k\Delta x. \end{aligned}$$

Taking the limit, the exact total mass is expressed by an integral.

$$\text{(Total mass)} \quad = \quad \int_a^b A(x)\rho(x)dx$$

The main challenge to these problems, naturally, is finding:

- The slicing variable (in this case x) so that density is nearly constant on slices, and
- A function $A(x)$ so that the volume of a slice is approximately $A(x)\Delta x$.

The form of the density function will, more often than not, suggest a good slicing variable.

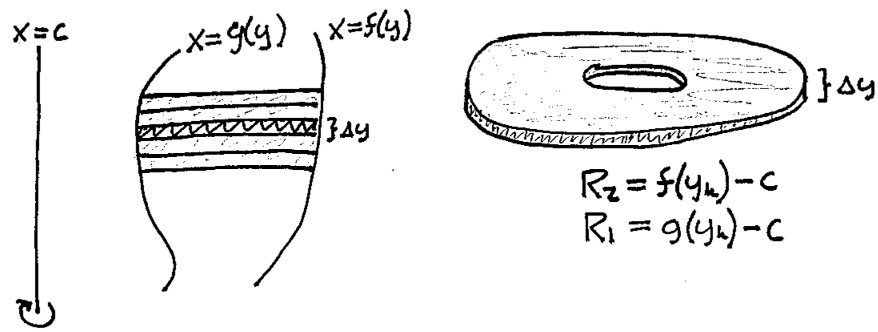
As mentioned in the introduction, I reiterate here that if the density function is constant, then this problem is identical to the problem of computing the volume of the shape.

$$\text{Density } \rho = 1 \text{ everywhere} \Rightarrow \text{(Total mass)} = \text{(Volume)}.$$

3 Solids of revolution

As one family of examples, we show here how the total mass of a substance in a volume of revolution can be found in many cases. When we want the total mass of a substance in a solid of revolution, we can slice the solid in the same ways as we did to find the volume, as long as this slicing gives slices where the density is nearly constant. This requirement on the density means that whereas to compute volume we could perhaps use shells or washers indifferently, to compute density one option or the other may be forced upon us. In particular, if the density varies with the distance to the axis, then washers cannot be used; we must use shells.

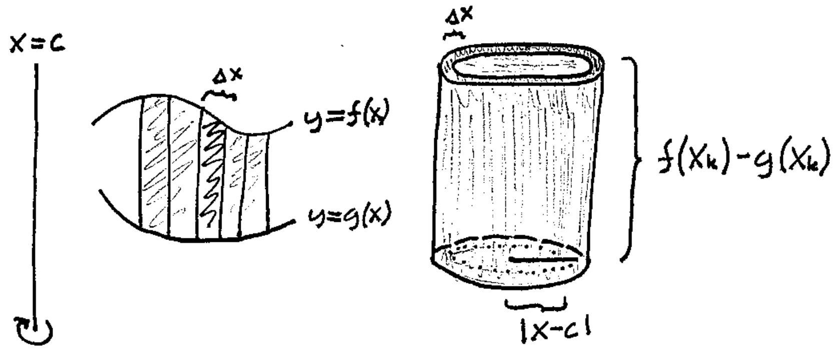
First consider slicing a solid using washers. Suppose that the solid is created by revolving the region between $x = g(y)$ and $x = f(y)$ around the axis $x = c$ (where $c \leq g(y) \leq f(y)$). As before, the region can be sliced into washers.



Then **as long as the density is nearly constant on these washers**, this same slicing scheme can be used to compute the total mass of the substance in the solid. If the density is given by $\rho(y)$, then the total mass is

$$\text{(Total mass)} = \int_a^b \pi \left[(f(y) - c)^2 - (g(y) - c)^2 \right] \rho(y)dy$$

Now consider slicing using cylindrical shells. Suppose that the solid is formed by revolving the region between the curves $y = f(x)$ and $y = g(x)$ for x in $[a, b]$ is revolved around $x = c$ (where $c \leq a$ and $f(x) \geq g(x)$).

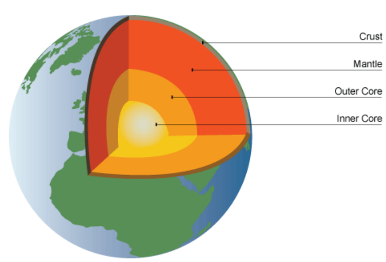


Then **as long as the density is nearly constant on these shells**, this same slicing scheme can be used to compute the total mass of the substance in the solid. In other words, the necessary assumption is that **density depends on the distance to the axis.** In this case, there is a function $\rho(r)$, which tells the density of the substance at a distance r from the axis. Then approximating the volume of each cylinder as usual and simply multiplying these volumes by the function $\rho(x-c)$ (since $x-c$ is the distance to the axis), the total mass is

$$\text{(Total mass)} = \int_a^b 2\pi(x-c)(f(x)-g(x))\rho(x-c)dx$$

4 Concentric slicing of a sphere

An important type of calculation in many situations is computing a total mass, where density varies based on a distance to a single point. For example, the mass of a spherical object such as a planet may be given by a density function that varies based on distance to the center.



Suppose that the solid is a sphere of radius R, and the density of a substance is given by $\rho(r)$, where r is the distance to the center. Then to slice with constant density, the sphere should be sliced into a sequence of concentric spheres. Suppose that the interval $[0, R]$ is divided into some number of pieces. Then the piece with larger endpoint r corresponds to a very thin spherical shell with thickness Δr and surface area $4\pi r^2$. The volume of this shell is then approximately $4\pi r^2 \Delta r$ (to see this, imagine the spherical shell as the coat of paint on a sphere; the amount of paint used should be the area of the surface of the sphere times the thickness of the paint). Therefore the amount of mass in this spherical shell is given by $4\pi r^2 \rho(r)\Delta r$. Therefore:

$$
\begin{aligned}
\text{(Total mass)} &\approx \sum_{k=1}^{n} 4\pi r_k^2 \rho(r_k)\Delta r \\
&\quad \text{(where } \Delta r = R/n,\ r_k = k\Delta r) \\
&= \int_0^R 4\pi r^2 \rho(r) dr
\end{aligned}
$$

Note of course that if $\rho(r) = 1$ for all R, then we recover from this the usual formula $\frac{4}{3}\pi R^3$ for the volume of a sphere.