An example of numerical approximation

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15 September 2011

Consider example 26.3 from Gottlieb's textbook. This example considers the integral $\int_1^5 \frac{dx}{x}$, and presents the values of the numerical approximations M_n, T_n, and S_{2n}, for $n = 4, 8, 50$, and 100. I shall show how to perform these calculations, in the case $n = 50$. Note that you do not need to go through all of these steps in order to compute approximations with different numbers of slices; it is only necessary to replace 50 with 100 (for example) where is appears in the final expression.

Recall the definitions of these approximations, for a function $f(x)$ (in this example, $f(x) = 1/x$).

$$
\begin{aligned}
M_n &= \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x \\
T_n &= \sum_{k=1}^{n} \left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\Delta x \\
S_{2n} &= \frac{2}{3}M_n + \frac{1}{3}T_n
\end{aligned}
$$

In order to compute these sums in this particular case, it is necessary to make each part of the first two formulas explicit, so that it can be entered into a machine. The Simpson's rule approximation can be computed afterward, by simply adding $\frac{2}{3}$ of one result and $\frac{1}{3}$ of the other. First of all, we have decided on n.

$$n = 50$$

Next, specify the terms $\Delta x, x_{k-1}$, and x_k. Note that I am not reducing fractions in some cases; this is mainly so that it will be clear in the final form where "n" occurs (namely, it will be where "50" occurs).

$$
\begin{aligned}
\Delta x &= (\text{length of interval})/n \\
&= 4/50 \\
\\
x_0 &= (\text{left end of interval}) \\
&= 1 \\
\\
x_k &= (k \text{ increments of } \Delta x \text{ from the left end}) \\
&= 1 + k\Delta x \\
&= 1 + 4k/50 \\
\\
x_{k-1} &= 1 + 4(k-1)/50
\end{aligned}
$$

Performing all of these replacements, the sums take the following form. I am, quite literally, copying and pasting to produce these.

$$
\begin{aligned}
M_{50} &= \sum_{k=1}^{50} f\left(\frac{1+4(k-1)/50+1+4k/50}{2}\right)\cdot\frac{4}{50} \\
T_{50} &= \sum_{k=1}^{50} \left(\frac{f(1+4(k-1)/50)+f(1+4k/50)}{2}\right)\cdot\frac{4}{50}
\end{aligned}
$$

The final substation that must be made is for the function $f(x)$. In this case, the function is $f(x) = 1/x$, so I do this by looking at each place where f occurs, copying the text of its argument, and pasting it in for x in the expression $1/x$.

$$
\begin{aligned}
M_{50} &= \sum_{k=1}^{50} \frac{1}{\left(\frac{1+4(k-1)/50+1+4k/50}{2}\right)}\cdot\frac{4}{50} \\
T_{50} &= \sum_{k=1}^{50} \left(\frac{\frac{1}{(1+4(k-1)/50)}+\frac{1}{(1+4k/50)}}{2}\right)\cdot\frac{4}{50}
\end{aligned}
$$

Now both of these sums are completely explicit and ready to be computed. How to compute them will depend on your hardware. By way of example, here is the syntax to compute the two sums on a TI-83 graphing calculator. I don't know if the kids are still using these calculators, but presumably syntax on newer models is similar.

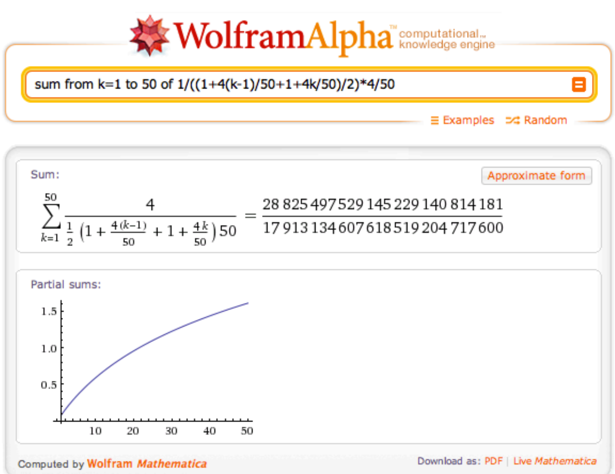
For the midpoint approximation M_{50}, type:

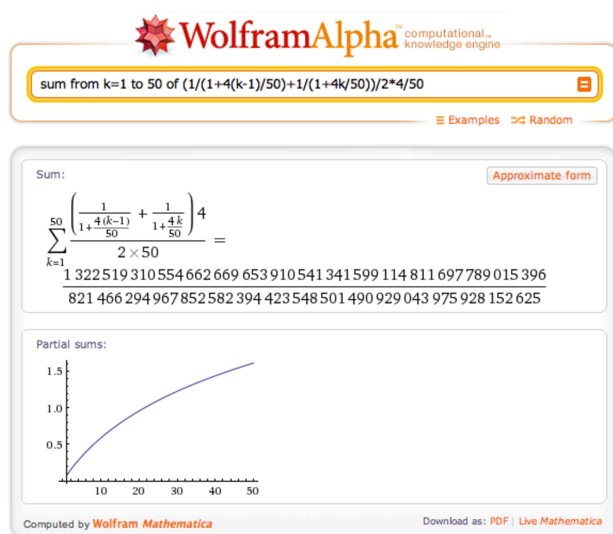
```
sum(seq(1/((1+4(K-1)/50+1+4K/50)/2)*4/50,K,1,50))
```

For the trapezoid approximation T_{50}, type:

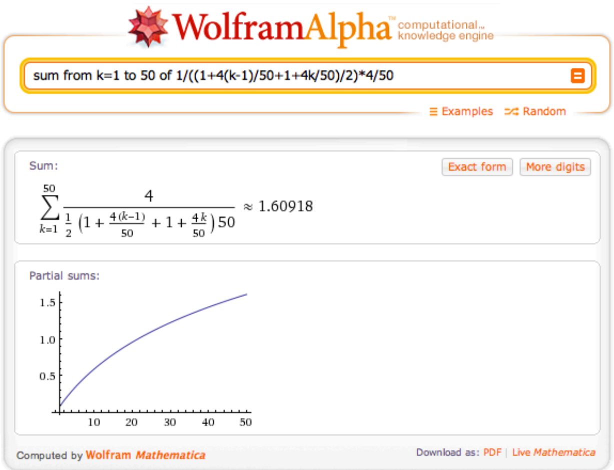
```
sum(seq(((1/(1+4(K-1)/50)+1/(1+4K/50))/2*4/50,K,1,50))
```

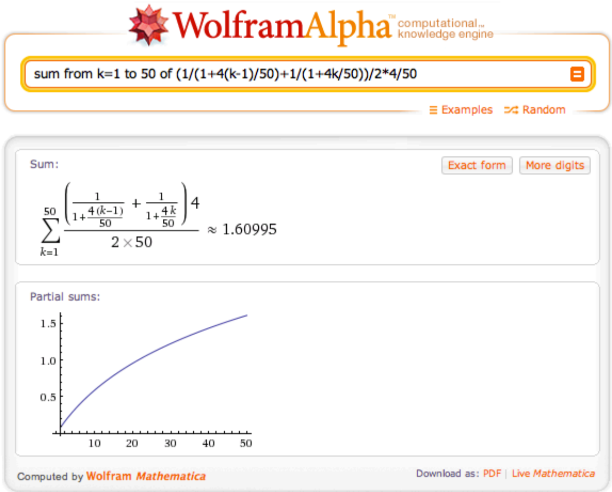
Alternatively, these can be computed online using Wolfram Alpha, by typing the phrase "sum from $k = 1$ to 50 of" and then the same syntax for the k^{th} term used above. Here are screenshots of the results.





Of course, the answer is not terribly useful as a massive fraction, so you should then press the "approximate form" button in the upper right. The result is shown.





It should be self-explanatory what to do if you want to see more digits of the value of the sum. Note that these values do indeed match the values given in Gottlieb's textbook.