Lecture 7: The derivative as a function

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1 Introduction

In this lecture, we introduce the idea that you can bundle together all the numbers $f'(c)$ (as c varies) into a *function* $f'(x)$, called the derivative of f. The graph of this function tells important qualitative and quantitative information about $f(x)$. The graphs of $f'(x)$ and $f(x)$ *look quite different*, but that is precisely why this transformation is so useful: it introduces a new perspective and new way to visualize.

We will discuss how to determine the derivative function visually, and also the formal way to derive it symbolically using limits. In later classes we'll develop a toolkit of "shortcut rules" that will make this process quicker in practice. The reference for today is Stewart §2.7.

2 Differentiation

We saw last time how to compute, given a function $f(x)$ and a constant c, a quantity called the derivative at the value $x = c$, denoted $f'(c)$. For example, if $(x) = x^2$, then for any number c, $f'(c) = \lim_{x \to c} \dfrac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c$. The basic idea of this lecture is that since you can compute this number at any value c you like, you can consider the *derivative function*. In this case, since $f'(c) = 2c$ for all numbers c, the *derivative function* is just the function $f'(x) = 2x$.

From any function $f(x)$, we obtain a new function $f'(x)$. This new function is called the derivative of $f(x)$. The process of determining a function's derivative is sometimes called *differentiation*. The origin of this word comes from the fact that the derivative essentially measures very small differences in nearby function values.

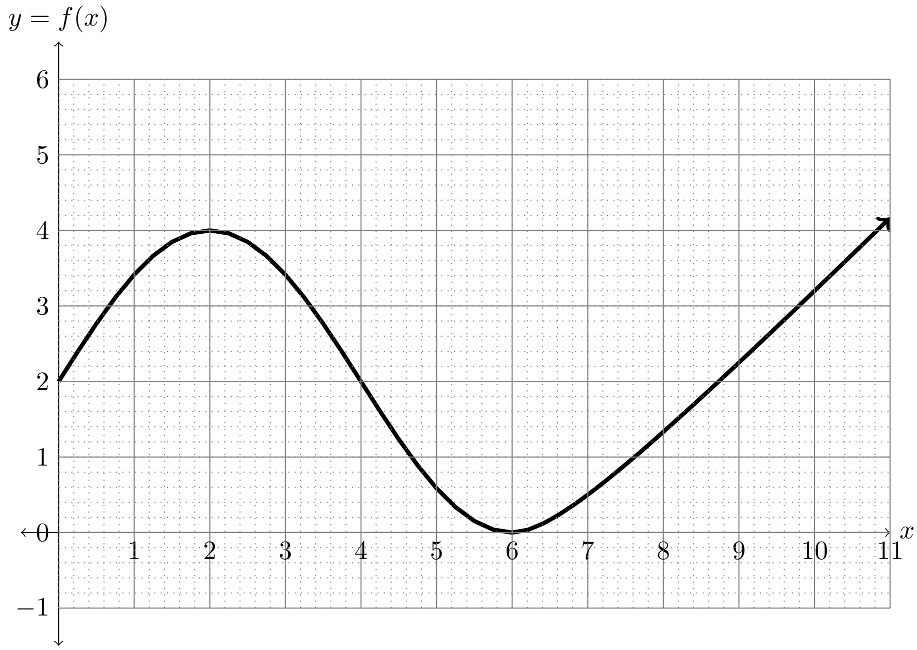
The following notation is sometimes also used for the derivative $f'(x)$. All these notations mean the same thing; you can use them interchangeably.

$$f'(x)$$
$$\frac{df}{dx}$$
$$\frac{d}{dx} f(x)$$

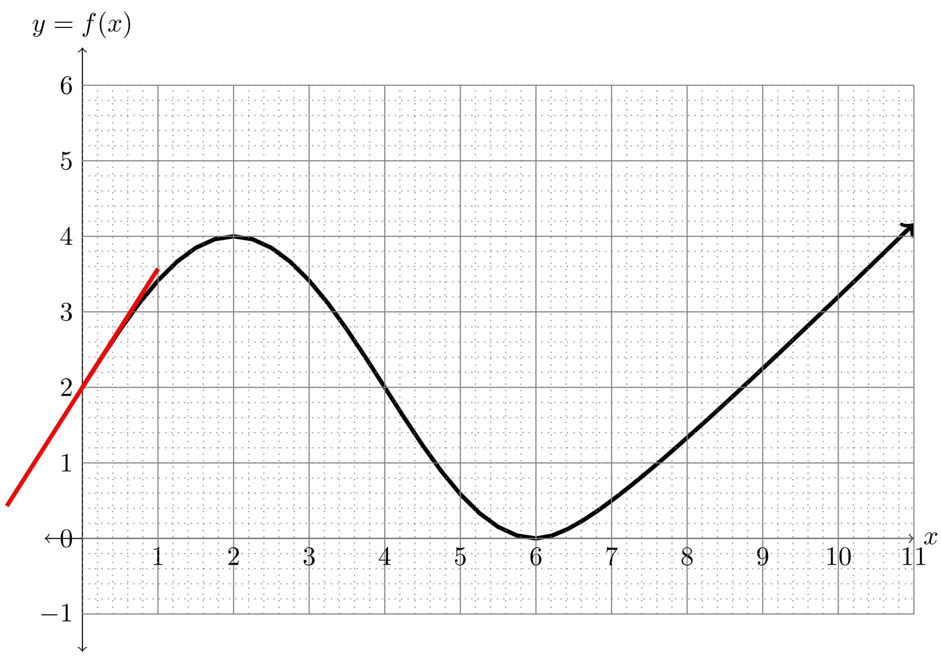
When a function is defined as a graph, e.g. by writing $y = x^2$ rather than $f(x) = x^2$, then it is also common to write $\frac{dy}{dx}$ for the derivative (in this case, $\frac{dy}{dx} = 2x$).

Aside. The "fraction" notation $\frac{df}{dx}$ is often called *Leibniz notation*, after one of the two founders of calculus who preferred writing derivatives this way (the other founder being Newton). It is designed to be intuitive; it should resemble the notation $\frac{\Delta f}{\Delta x}$; the idea is that the derivative is like the ration of the change in f to the change in x (near a particular point), except that both these changes are infinitely small (infinitesimal). The symbols df and dx don't have meaning all by themselves as far as we are concerned (they have very definite and precise meanings in more advanced topics in math); you should think of them as being metaphorical in this course. They are sometimes called *differentials.* At one time they were derisively called *ghosts of departed quantities.*

Example 2.1. Consider the function $f(x)$, defined for positive numbers x, shown in the following graph. Sketch the graph of the function $f'(x)$.

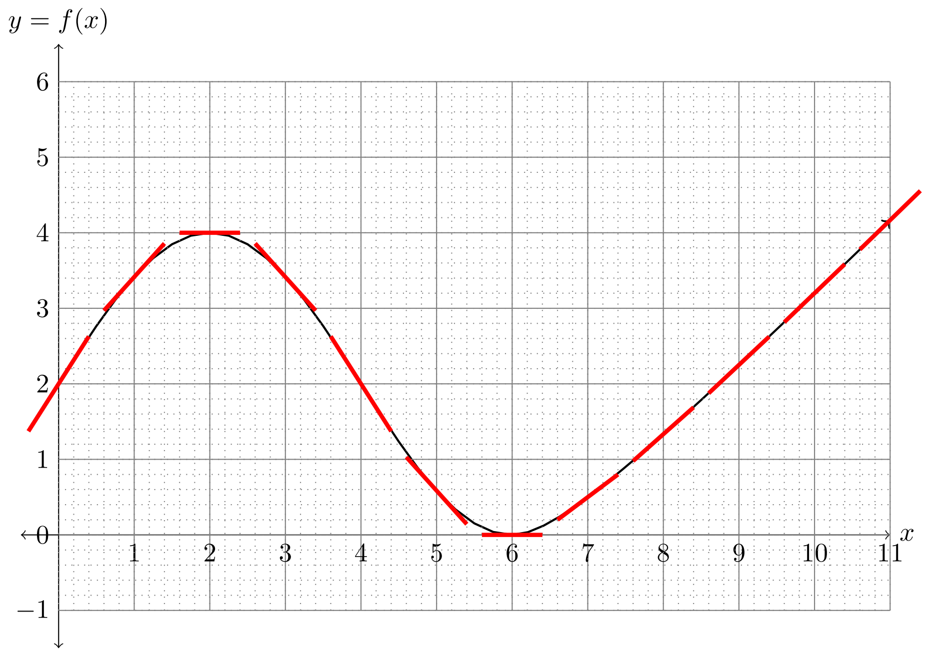


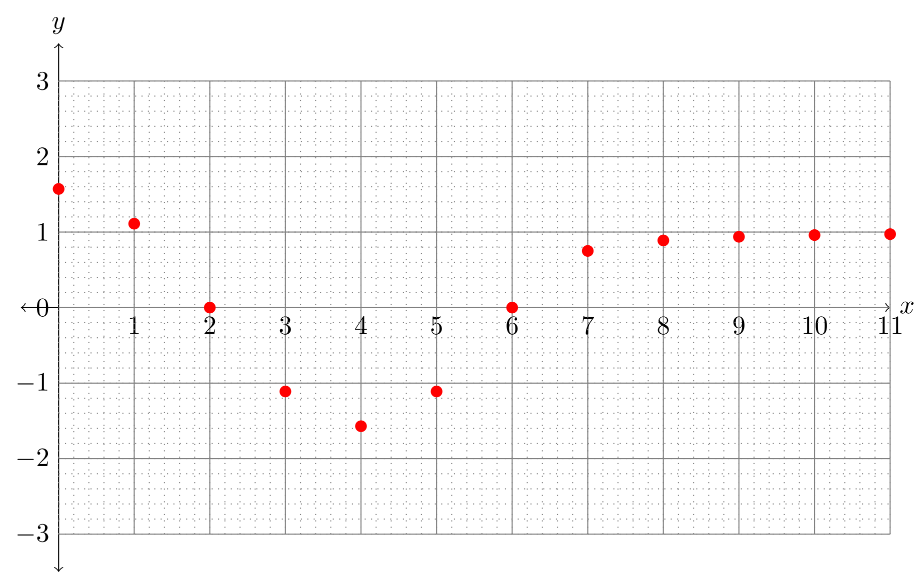
Solution. Remember that $f'(c)$ is the slope of the tangent line to the graph at the point $(c, f(c)$. So to sketch the graph, we can measure $f'(c)$ at several values of c by carefully drawing tangent lines to the graph. To begin, draw a tangent line to the graph at $x = 0$.



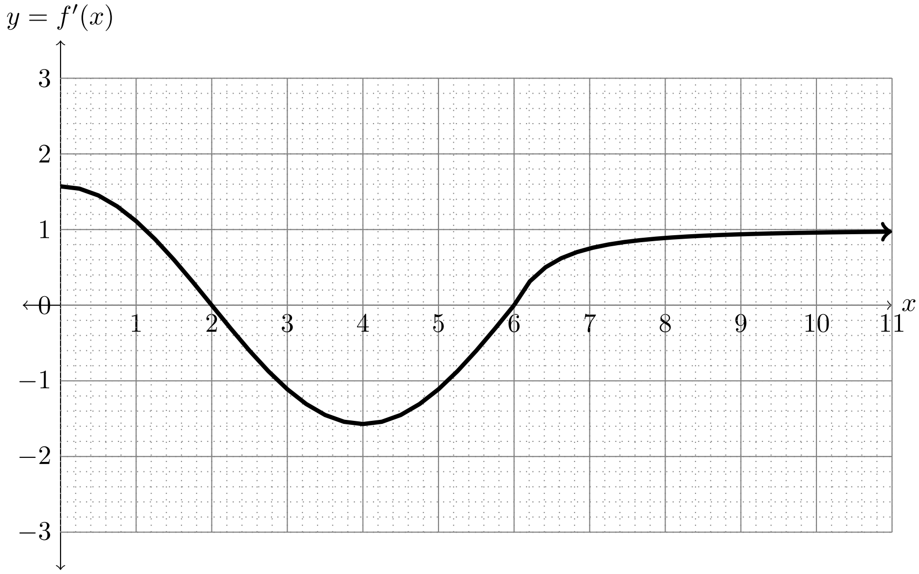
If you use the grid lines to measure the slope of this line, you will see that it is very close to 1.5 (pick any two points on the line, measure the difference in y values, and divide by the difference in x values).

Continuing in this way, we can draw a tangent line at each point $x = 0, 1, 2, \cdots, 11$, measure the slope, and plot the resulting points, as shown below.





These data points give a pretty good notion of what the plot of $f'(x)$ looks like. It is shown below.



There are two important features to notice in this example: first, $f(x)$ is increasing precisely where $f'(x) > 0$, and $f(x)$ is decreasing precisely where $f'(x) < 0$. Where $f'(0) = 0$, the graph of $f(x)$ appears to lie flat.

3 Finding the derivative function with limits

Finding the derivative function using limits is no more complicated that finding it at a point, like we did last time. There are two main methods to do it; they are basically equivalent, but they look a little bit different. You may find one or the other easier to carry out, and both are equally acceptable. Most students find method 2 easier.

For a very simple example, here are both methods, used to differentiate $f(x) = 5x + 7$.

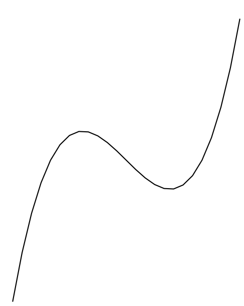
Method 1	Method 2
$f(x) = 5x + 7$	$f(x) = 5x + 7$
$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{z \to x} \frac{(5z + 7) - (5x + 7)}{z - x}$	$= \lim_{h \to 0} \frac{(5(x+h) + 7) - (5x + 7)}{h}$
$= \lim_{z \to x} \frac{5z - 5x}{z - x}$	$= \lim_{h \to 0} \frac{5h}{h}$
$= \lim_{z \to x} 5$	$= \lim_{h \to 0} 5$
$= 5$	$= 5$
$f'(x) = 5.$	$f'(x) = 5$

So the derivative of $f(x) = 5x + 7$ is the constant function $f'(x) = 5$. One thing to notice here is that the "+7" is totally irrelevant; it cancels immediately before taking the limit. In general, adding a constant does not affect the derivative of a function.

Both these methods are essentially the same. The second simply replaces the variable z with $x+h$, where h is another variable. Both have the same basic form: a limit of slopes of secant lines. The times when one *might* be slightly more convenient than another are times when it's easier to simplify the expression.

Aside. You may wonder why I'm suddenly using the variable z above. I do this simply because the variable x is already taken, so a I need a new name for the function inputs that will be approaching x to compute the derivative. If you prefer, you could just compute the limit like we did last time (using $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$) and simply substitute x for x in the expression at the end.

Here's a more intricate example: consider the function $f(x) = x^3 - x$. Its graph looks something like this.



Before commuting the derivative symbolically, take a moment to imagine what it must look like. The function increases, then decreases, then increases. So the derivative must be a function that starts positive, then dips negative for some time, then becomes positive again. We'll see of course that this is the case.

Here's how you begin to set up the limit.

Method 1	Method 2
$f(x) = x^3 - x$	$f(x) = x^3 - x$
$f'(x) = \lim\limits_{z \to x} \dfrac{(z^3 - z) - (x^3 - x)}{z - x}$	$f'(x) = \lim\limits_{h \to 0} \dfrac{\big((x+h)^3 - (x+h)\big) - (x^3 - x)}{h}$
$= \lim\limits_{z \to x} \dfrac{z^3 - x^3 - z + x}{z - x}$	$= \lim\limits_{h \to 0} \dfrac{(x+h)^3 - x^3 - h}{h}$

At this point, both methods require some insight from algebra. For method one, you need the following factorization of a difference of two cubes: $z^3 - x^3 = (z - x)(z^2 + zx + x^2)$. For method two, you need to know how to expand $(x + h)^3$. A foolproof way to do this is to first expand $(x + h)^2$, then multiply it by $(x + h)$, as follows.

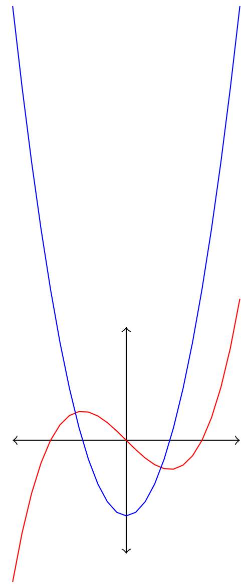
$$
\begin{aligned}
(x+h)^3 &= (x+h)^2(x+h) \\
&= (x^2 + 2xh + h^2)(x+h) \\
&= (x^2 + 2xh + h^2)x + (x^2 + 2xh + h^2)h \\
&= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 \\
&= x^3 + 3x^2h + 3xh^2 + h^3
\end{aligned}
$$

(Note: on homework, you could skip to the last step here; I reproduce the details merely in case they are not familiar.)

Using one insight or the other, we can now continue the analysis as follows.

Method 1 (continued)	Method 2 (continued)
$f'(x) = \lim\limits_{z \to x} \dfrac{(z-x)(z^2 + zx + x^2) - (z - x)}{z - x}$	$f'(x) = \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3} - h}{h}$
$= \lim\limits_{z \to x} \big(z^2 + zx + x^2 - 1\big)$	$= \lim\limits_{h \to 0} 3x^2 + 3xh + h^2 - 1$
$= x^2 + x^2 + x^2 - 1$	$= 3x^2 + 0 + 0 - 1$

So in both cases, we see that $\boxed{f'(x) = 3x^2 - 1}$. It is instructive to look at the graphs of both $f(x)$ and $f'(x)$ on the same axes. Here $f(x)$ is shown in red and $f'(x)$ is shown in blue.



I'll finish this subsection with one more somewhat intricate example. The technique is the same as above, although it requires several key techniques for manipulating fractions and square roots.

Example 3.1. Let $f(x) = \frac{1}{\sqrt{x}+1}$. Determine $f'(x)$.

Solution.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}+1} - \frac{1}{\sqrt{x}+1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(\sqrt{x}+1)-(\sqrt{x+h}+1)}{(\sqrt{x+h}+1)(\sqrt{x}+1)}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x+h}+1)(\sqrt{x}+1)} \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x+h}+1)(\sqrt{x}+1)} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h\to 0} \frac{x-(x+h)}{h(\sqrt{x+h}+1)(\sqrt{x}+1)\left(\sqrt{x}+\sqrt{x+h}\right)} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{x+h}+1)(\sqrt{x}+1)\left(\sqrt{x}+\sqrt{x+h}\right)} \\
&= \frac{-1}{(\sqrt{x}+1)\cdot(\sqrt{x}+1)\left(\sqrt{x}+\sqrt{x}\right)} \\
&= -\frac{1}{2(\sqrt{x}+1)^2\sqrt{x}}
\end{aligned}$$

So the derivative function is $f'(x) = -\frac{1}{2(\sqrt{x}+1)^2\sqrt{x}}$.

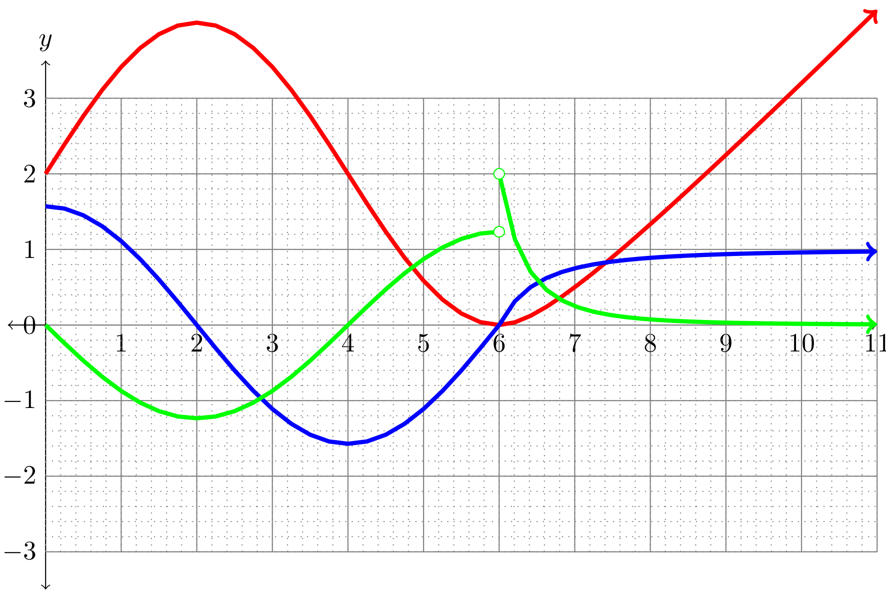
4 The second derivative

You can differentiate any function. In particular, you can differentiate a function that is already the derivative of some other function. This is called the second derivative, and it is often denoted as $f''(x)$, or using "Leibniz notation" as $\frac{d^2}{dx^2}f(x)$ (personally I find Leibniz notation confusing for second derivatives, though).

Example 4.1. Let $f(x) = x^2$. Then as we've seen before, $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x$. Therefore $f''(x) = \lim_{h\to 0} \frac{2(x+h) - 2x}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2$. So if $f(x) = x^2$, then $f'(x) = 2x$ and $f''(x) = 2$.

Example 4.2. **A flying object.** One of the most basic principles of physics that Newton first expressed using the new language of calculus was that gravity causes an object in midair (not being propelled by anything else) to accelerate down towards the earth at a *constant rate*. This explains why a thrown ball travels in a parabolic path: parabolas are graphs of functions with linear velocity and constant acceleration.

Example 4.3. Consider the function $f(x)$ studied in the first section. We found a graph of its derivative by plotting a sequence of points and drawing a curve through them. You could in principle do the same thing to obtain the second derivative (though you'd want to do it very accurately to get a good curve). Shown below is the function $f(x)$ (red), its derivative $f'(x)$ (blue) and its second derivative $f''(x)$ (green).



Look carefully at the blue and green graphs and convince yourself that one is the derivative of the other. The fact that the second derivative (green graph) has a jump discontinuity at $x = 6$ is difficult to see from the picture of the blue graph. Don't worry too much about why this happens for now; we'll discuss the phenomenon more in the next lecture (it is an example of a *corner*).