Lecture 34: Integration with density

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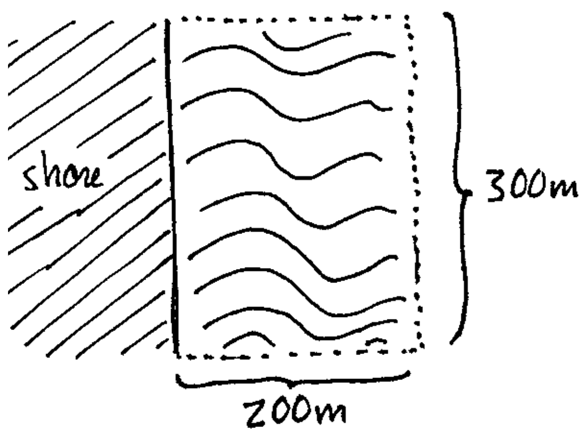
1 Introduction

We will finish this course with one more example of a physical situation you can transform into an integral, using the idea of slicing. This is the problem of measuring a quantity that is unevenly distributed across some region, given information about its density at various parts of the region.

Rather than go into too much generality, I will just present three representation examples of this sort of problem. It is covered in somewhat more depth in Math 1B.

2 Density across a rectangle

Consider a 200 by 300 meter patch of ocean, as shown.



Unfortunately for you, this patch of water is full of rogue, human-eating sharks! The sharks are densest right near the shore (where they are close to the people), and the sharks become more sparse further out at sea. In fact, if x denotes the distance from the shore to a given point, then the density of sharks around that point is given by some function:

$$\text{shark density } = \rho(x) \text{ sharks}/m^2$$

How can we compute the total number of sharks, from this density function?

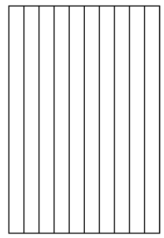
The basic idea is this: if the sharks had *constant* density, we could compute the number by simple multiplication.

$$\text{number of sharks} = (\text{area}) \cdot (\text{density of sharks})$$

Check the units: area is expressed in m^2, shark density is in sharks per m^2, so their product will be expressed in sharks.

Now, this formula does work for this problem: the shark density is different at different places. However, there is a good way to deal with this situation: **slice the region into small rectangles, where the shark density is nearly constant in each rectangle**.

So how should we slice? The shark density only depends on the distance x to shore, so slice into vertical rectangles: this way x will not vary much within each rectangle.



Now the shark density is pretty close to constant in each rectangle. As usual, lets define the x values dividing these rectangles as follows. The number n is the number of rectangles. We choose Δx to be one nth of the difference from the smallest x value to the largest – here the smallest is 0 and the largest is 200 – and then let x_k be the smallest plus k times Δx. See the lecture on Riemann sums for more discussion on this procedure.

$$\begin{aligned} \Delta x &= 200/n \\ x_k &= k \cdot \Delta x \end{aligned}$$

Now the kth rectangle has x value varying from x_{k-1} to x_k; these differ by only Δx, and if n is large enough this difference won't be much: so the density throughout the whole rectangle is pretty close to $\rho(x_k)$. Therefore we have the following approximating for the number of sharks in the kth slice (which gets better and better as n gets larger and larger).

$$\begin{aligned} \#\text{sharks in slice } k &\approx (\text{area of slice } k) \cdot \rho(x_k) \\ &\approx 300\Delta x \cdot \rho(x_k) \\ &\approx 300\rho(x_k) \cdot \Delta x \end{aligned}$$

Therefore, to approximate the total number of sharks, just add all these quantities up.

$$\begin{aligned} \#(\text{sharks}) &= \sum_{k=1}^{n} \#(\text{sharks in slice } k) \\ &\approx \sum_{k=1}^{n} 300\rho(x_k) \cdot \Delta x \end{aligned}$$

Now, notice that this has exactly the form of a Riemann sum. So to compute the exact number of sharks, just turn it into an integral.

$$\#\text{sharks} = \int_0^{200} 300\rho(x)dx$$

Of course, the value of this integral will depend on the function $\rho(x)$. Different values of $\rho(x)$ will give different counts of sharks. The value of the integral expression is that it shows exactly how the total depends on the (varying) density.

Example 2.1. Suppose for concreteness that the density is given by $\rho(x) = 2^{-x/50}$. Take a moment to think about what this means physically: it means that at $x = 0$ (at the shore, or close to it), there are $\rho(0) = 1$ sharks per square meter. So they are quite crowded indeed! Now, for every 50 meters you swim out to sea (if you can make it), $\rho(x)$ decreases by a factor of 2. At 50 meters out the sharks have thinned to 0.5 sharks per square meter (or 1 shark per two square meters), and once you swim all the way out $200m$ they have thinned to one for every 16 square meters.

If this is the case, how many sharks are there total? We just compute the integral:

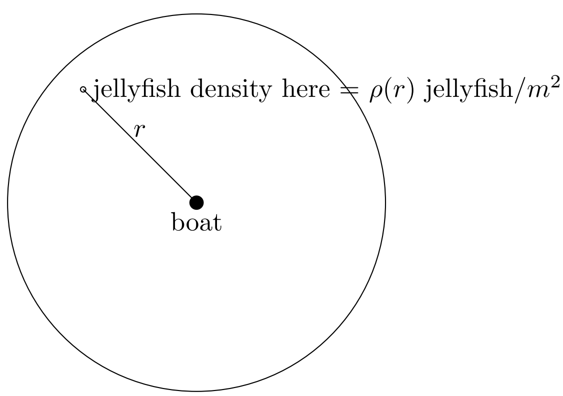
$$\begin{aligned}
\#\text{sharks} &= \int_0^{200} 300 \cdot 2^{-x/50} dx \\
&= \left[300 \cdot \frac{-50}{\ln 2} 2^{-x/50}\right]_0^2 00 \\
&= \frac{15000}{\ln 2}\left(-2^{-4}+1\right) \\
&= \frac{28125}{\ln 2} \\
&\approx 20,288.
\end{aligned}$$

Notice that the total area of ocean is 60000 square meters, so this suggests that on average there is one shark per 3 square meters, which is plausible since the local density ranges from one per square meter to one per 16 square meters.

3 Slicing into non-rectangles

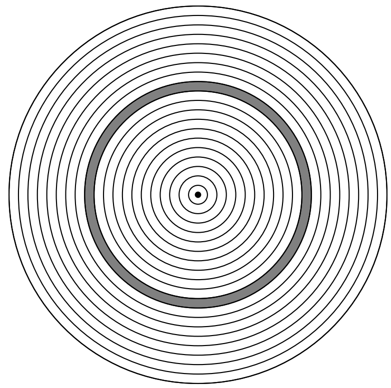
With density problems, your hand is always forced in terms of how you must slice: you have to slice so that the density is almost constant on each slice. That can mean slicing into small pieces other than rectangles. For example, consider the following situation.

Suppose that you are sitting on a boat, putting some sort of jellyfish food into the water. As a result, a crowd of jellyfish is forming around your boat. They are densest near the boat, and they become less dense the further you travel from the boat. Let's say that the density of the jellyfish is a function of distance r from the boat, $\rho(r)$.



We would like to compute the total number of jellyfish **within** 1000 **meters of the boat**.

How can we express the total number of jellyfish with an integral? We begin by slicing this patch of ocean not into rectangles, but into little rings. The reason is that jellyfish density is roughly constant on these rings, so we can count jellyfish by multiplying density by area.



Each slice has the shape of a ring. If we divide all the r values (from 0 to 1000) into n subintervals, bounded by values $r_0, r_1, \cdots, r_n$, then the kth slice is a ring whose outer radius is r_k and whose inner radius is r_{k-1}. On this ring, the jellyfish density is very close to $\rho(r_k)$, since it does not change much between r_{k-1} and r_k. Therefore

$$\#(\text{jellyfish in slice } k) \approx (\text{area of slice } k) \cdot \rho(r_k)$$

Now we just need to approximate the area of slice k. To do this, notice that the area of the ring should be about equal to the thickness times the circumference (you can think of the ring like a rectangle that's been bent around to link itself). Therefore:

$$\begin{aligned}(\text{area of slice } k) &\approx (\text{circumference}) \cdot (\text{thickness}) \\ &\approx 2\pi r_k \cdot \Delta r\end{aligned}$$

Note for pickier people. The exact area is actually $\pi r_k^2 - \pi r_{k-1}^2$, which you can express as $2\pi \cdot \frac{r_k + r_{k-1}}{2} \cdot \Delta r$ with a little algebra. One way the justify the approximation above is that $\frac{r_{k-1}+r_k}{2} \approx r_k$, since $r_{k-1} \approx r_k$.

From this, we obtain an estimate on the total number of jellyfish within 1000 meters.

$$\begin{aligned}\#(\text{jellyfish}) \quad &\approx \quad \sum_{k=1}^{n} 2\pi \cdot r_k \cdot \rho(r_k)\Delta r \\ \text{where} \quad &\Delta r = \frac{1000}{n},\ r_k = k \cdot \Delta r\end{aligned}$$

We can take the limit as n goes to infinity to obtain the following exact[1] expression for the number of jellyfish.

$$\#(\text{jellyfish}) = \int_0^{1000} 2\pi r\ \rho(r)\ dr$$

One thing to notice about this integral: the density $\rho(r)$ counts more when r is larger. This makes sense: the outer rings are bigger, so the density there has a greater effect.

To illustrate this expression, and also as an excuse for some integration review, I'll do two examples.

Example 3.1. Suppose that $\rho(r) = e^{-r^2/1000}$. So the jellyfish are spread about one per square meter right at the boat, but they significantly thin out further from the boat. In this case, we can compute the total number of jellyfish by substitution.

[1] "Exact" isn't really the right phrase here, since the density function itself won't be exact (the jellyfish don't spread their mass out evenly, they swim around). So really this is a sort of "idealized count;" one more suitable to apply the machinery and transformations that calculus allows.

$$
\begin{aligned}
\#(\text{jellyfish}) &= \int_0^{1000} 2\pi r\ e^{-r^2/1000} dr \\
& \quad u = -r^2/1000,\ du = -r/500 dr \\
&= \int_0^{-1000} 2\pi r e^u \frac{-500du}{r} \\
&= -1000\pi \int_0^{-1000} e^u du \\
&= 1000\pi \int_{-1000}^{0} e^u du \\
&= 1000\pi \left[e^u\right]_{-1000}^{0} \\
&= 1000\pi(1 - e^{-1000}) \\
&\approx 3142
\end{aligned}
$$

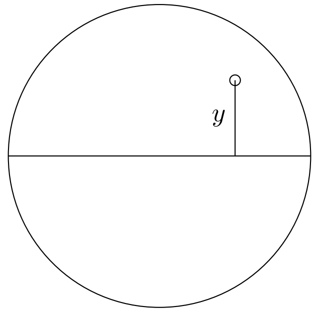
Example 3.2. Suppose that the jellyfish density still decays quickly, but not as quickly: $\rho(r) = e^{-r/1000}$. Then we can compute the number of jellyfish using a combination of substation and integration by parts. The result is much larger in this situation.

$$
\begin{aligned}
\#(\text{jellyfish}) &= \int_0^{1000} 2\pi r e^{-r/1000} dr \\
& \quad u = -r/1000,\ du = -\frac{1}{1000} dr \\
&= \int_0^{-1} 2\pi(-1000u)e^u \cdot (-1000) du \\
&= 2000000\pi \int_0^{-1} u e^u du \\
& \quad v = u,\ dw = e^u du \\
& \quad dv = du,\ w = e^u \\
&= 2000000\pi \left(\left[u e^u\right]_0^{-1} - \int_0^{-1} e^u du \right) \\
&= 2000000\pi \left(-e^{-1} - \left[e^u\right]_0^{-1} \right) \\
&= 2000000\pi(-e^{-1} - e^{-1} + 1) \\
&= 2000000\pi(1 - 2/e) \\
&\approx 1.6 \text{ million}
\end{aligned}
$$

4 Rectangles of irregular dimensions

The final example concerns a situation where we do slice into rectangles, with one dimension Δx, but the other dimension is irregular. The basic procedure is essentially similar.

Suppose that a pond 400 meters in diameter has a rope drawn across the middle. There are some lilly pads in the pond; their density depends on the distance y to the rope.



At the marked point, the lilly density is $\rho(y)$ lilly pads per square meter.

In this situation, slice the region into horizontal rectangles. The height of rectangle k will be Δy (like when we've sliced horizontally before), while the width now requires the Pythagorean theorem to calculate: the result is $2\sqrt{200^2 - y_k^2}$ (the two points on the circle with $y = y_k$ have $x = \pm\sqrt{200^2 - y_k^2}$, hence the different between these is twice the square root). Therefore, going straight to the integral:

$$\#(\text{lilly pads north of the rope}) \quad = \quad \int_0^{200} 2\sqrt{200^2 - y^2}\rho(y)dy$$

Finally, we can use **symmetry** to get the full count: there are exactly as many jellyfish south of the rope as north of it. So the total is as follows.

$$\#(\text{lilly pads}) = 4\int_0^{200} \sqrt{200^2 - y^2}\rho(y)dy$$