Lecture 32: Integration by parts

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1 Introduction

Integration by parts, similarly to integration by substitution, reverses a well-known technique of differentiation and explores what it can do in computing integrals. Like integration by substitution, it should really be viewed not necessarily as a means of solving an integral, but as a means of transforming an integral. Whether this transformation produces a more tractable integral integral or so will depend on how it is applied.

For our purposes, this technique is mainly included to expand the collection of antiderivatives that will be familiar to you upon completing the course. As a secondary objective, a good qualitative understanding of the technique sheds a great deal of light on other parts of the subject. Integration by parts is a critical technique in understanding various phenomena in electricity and magnetism, for example, precisely because of the sort of transformation it induces. In the optional appendix, section 6, I indicate a geometric way to view integration by parts, which has been far more important for me, both in mathematical work and in studying other subjects, than any calculations of any particular integrals.

The reference for today is Stewart §5.6.

2 The technique in general

Integration by parts is simply the product rule in reverse. Suppose that $u(x), v(x)$ are two different functions of x. Then the product rule states that

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x).$$

To obtain the integration version of this, integrate both sides with respect to x to obtain the following, which is usually written with the arguments "(x)" suppressed, in differential notation.

$$\begin{aligned} u(x)v(x) &= \int v(x)u'(x)dx + \int u(x)v'(x)dx \\ uv &= \int vdu + \int udv \end{aligned}$$

Here, we are regarding u and v as functions of x despite suppressing the argument, and du, dv are the usual shorthand, denoting respectively $u'(x)dx$ and $v'(x)dx$. The usual way of expressing integration by parts is the following.

$$\boxed{\int udv = uv - \int vdu}$$

Integration by parts may be understood as follows: it transforms an integral by differentiating one part and integrating another part. This explains the transition from $\int udv$ to $\int vdu$. The minus sign and the term uv are by-products of this transition. The reason to perform this transformation is usually that the integral $\int vdu$ is somehow easier than the original, but there may be other reasons, which are illustrated in the examples.

Integration by parts is most likely to help in situations where u becomes much simpler upon differentiation. This is especially true if it is a polynomial. Other good candidates are exponential functions, sines, and cosines.

Often, it is necessary to perform integration by parts multiple times. This is especially the case when u is a polynomial (such as x^2). There is a systematic way to do this and keep track of all the signs, which is described in section 5.

2.1 Definite integrals

To evaluate definite integrals, you can either compute the indefinite integral and then plug in limits, or you can track the limits of integration as you go. For the second option, you will end up with an expression like the following.

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)dx$$

Alternative, if you wish to evaluate the integral $\int vdu$ with respect to the variable u, rather than x, the following is also valid. Since I am now suppressing the arguments "(x)", it is necessary to specify which variable is going from a go b explicitly in the subscripts and superscripts.

$$\begin{aligned}\int_{x=a}^{x=b} udv &= [uv]_{x=a}^{x=b} - \int_{x=a}^{x=b} vdu \\ &= [uv]_{x=a}^{x=b} - \int_{u(a)}^{u(b)} vdu\end{aligned}$$

In the second line, I have converted the limits of integration from values of x to values of u. This will only be a useful thing to do if v can be written as a function of u, and thus the second integral written as an integral with respect to u. For an example of where this can be done, see examples 3.3 and 3.7. This is also a useful viewpoint in the optional appendices 6 and 7.

3 Examples

Example 3.1. $\displaystyle\int xe^x dx$

This is a typical case: the e^x is easy to integrate, and the x will become simpler if it is differentiated.

$$\begin{aligned}\int xe^x dx & & u = x,\ v = e^x dx \\ & & du = dx,\ v = e^x \\ &= & xe^x - \int e^x dx \\ &= & xe^x - e^x + C\end{aligned}$$

Example 3.2. $\displaystyle\int_0^{\pi/2} x\cos x dx$

This is very similar to the example 3.1: target the x for differentiation, since it becomes much simpler (1), while the cosine doesn't become any more complicated under integration.

$$\begin{aligned}
\int_0^{\pi/2} x\cos x dx & \qquad u = x,\ dv = \cos x dx \\
& \qquad du = dx,\ v = \sin x \\
&= [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \\
&= \frac{\pi}{2} - [-\cos x]_0^{\pi/2} \\
&= \frac{\pi}{2} - 1
\end{aligned}$$

Example 3.3. $\int_1^e \ln x dx$

This integral is one of the classics of integration by parts. The idea behind it is the eliminate the logarithm by taking its derivative and producing the much simpler $\frac{1}{x}$. The price paid for this is the integral of dx which is simply x, and in fact this is quite a small price.

$$\begin{aligned}
\int_1^e \ln x dx & \qquad u = \ln x,\ dv = dx \\
& \qquad du = \frac{1}{x}dx,\ v = x \\
&= [x\ln x]_1^e - \int_1^e x \cdot \frac{1}{x} dx \\
&= [x\ln x]_1^e - \int_1^e dx \\
&= [x\ln x - x]_1^e \\
&= (e-e)-(0-1) \\
&= 1
\end{aligned}$$

Example 3.4. 1. $\int x^2 e^x dx$

$$\begin{aligned}
\int x^2 e^x dx & \qquad u = x^2,\ dv = e^x dx \\
& \qquad du = 2xdx,\ v = e^x \\
&= x^2e^x - \int (2x)e^x dx \\
&= x^2e^x - 2\int xe^x dx \\
&= x^2e^x - 2xe^x + 2e^x
\end{aligned}$$

At the last stage, we have used the result of the first exercise.

2. $\int x^3 e^x dx$

$$\begin{aligned}
\int x^3 e^x dx \qquad & u = x^3,\ dv = e^x dx \\
& du = 3x^2 dx,\ v = e^x \\
= \quad & x^3 e^x - \int 3x^2 e^x dx \\
= \quad & x^3 e^x - 3(x^2 e^x - 2xe^x + 2e^x) + C \\
= \quad & = e^x(x^3 - 3x^2 + 6x - 6) + C
\end{aligned}$$

Here, again, we have used the result of the previous part. Note that in each case, the integral is transformed, by integration by parts, into the same integral, with a smaller exponent for x.

3. $\int x^n e^x dx$
By using $u = x^n,\ dv = e^x dx$, the following "reduction formula" is obtained.

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

By repeated application of this formula, any such integral can be calculated. The result may be written as follows.

$$\int x^n e^x dx = e^x(x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \cdots + (-1)^n(n!)) + C$$

Example 3.5. 1. $\int x \ln x dx$
If $\ln x$ is differentiated, then what remains will be only powers of x, so it seems most profitable to perform the substitution $u = \ln x$.

$$\begin{aligned}
\int x \ln x dx \qquad & u = \ln x,\ dv = x dx \\
& du = \frac{1}{x} dx,\ v = \frac{1}{2} x^2 \\
= \quad & \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x dx \\
= \quad & \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C
\end{aligned}$$

2. $\int x^n \ln x dx$
Try the same tack as in the previous problem.

$$\begin{aligned}
\int x^n \ln x dx \qquad & u = \ln x,\ dv = x^n dx \\
& du = \frac{1}{x} dx,\ v = \frac{1}{n+1} x^{n+1} \\
= \quad & \frac{1}{n+1} x^{n+1} \ln x - \int \frac{1}{n+1} x^n dx \\
= \quad & \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C
\end{aligned}$$

Example 3.6. $\int (\ln x)^2 dx$

There are at least two approaches that will work here. One is to apply integration by parts immediately, using $u = (\ln x)^2,\ dv = dx$. This will reduce the problem to integrating $\ln x dx$, which is an integral that is already known.

Another approach is to first perform a substation to eliminate the logarithm, as follows.

$$\begin{aligned} \int (\ln x)^2 dx & \qquad u = \ln x,\ du = \frac{1}{x}dx \\ &= \int u^2 x du \\ &= \int u^2 e^u du \end{aligned}$$

Now this has been transformed to the integral from the first part of example 3.4. Hence using the result of that example, the result is

$$\begin{aligned} \int (\ln x)^2 dx &= e^u(u^2 - 2u + 2) + C \\ &= x((\ln x)^2 - 2\ln x + 2) + C \end{aligned}$$

Example 3.7. $\int \tan^{-1} x dx$

This is another case where the derivative of the integrand is of a somewhat simpler form than the original integrand, so taking it for u will help.

$$\begin{aligned} \int \tan^{-1} x dx & \qquad u = \tan^{-1} x,\ dv = dx \\ & \qquad du = \frac{1}{1+x^2}dx,\ v = x \\ &= x\tan^{-1} x - \int \frac{x}{x^2+1}dx \end{aligned}$$

This new integral now can be evaluated by means of a substitution for $x^2 + 1$, yielding the following result.

$$\int \tan^{-1} x = x\tan^{-1} x - \frac{1}{2}\ln|x^2+1| + C$$

This integral, as well as that of $\ln x$ and a number of others, fits into a general scheme which can be used to evaluate integrals of many inverses of well-understood functions. This is discusses in more detail in section 7.

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4 The last example from lecture 31

Integration by parts allows us to complete the last example from lecture 31, which was $\int \cos(\sqrt{x})dx$. The substation $u = \sqrt{x}$ gave the following re-expression.

$$\int \cos(\sqrt{x})dx = \int 2u\cos u du$$

Here, we know that the u will become simpler by differentiation, and that the $\cos u$ will become no more complicated by integrating. So take $v = u$ and $dw = \cos u du$ to proceed (here I use v and w for my integration by parts variables, since u is already taken).

$$\begin{aligned}
\int 2u \cos u du \qquad & v = 2u,\ dw = \cos u\ du \\
& dv = 2du,\ w = \sin u \\
= \quad & 2u \sin u - \int 2 \sin u du \\
= \quad & 2u \sin u + 2 \cos u + C
\end{aligned}$$

Therefore, back-substitution $u = \sqrt{x}$ finally gives the answer.

$$\int \cos(\sqrt{x}) dx \quad = \quad 2\sqrt{x} \sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$$

5 Tabular Integration

This section should be considered optional, although it may be useful in keeping track of notation while performing integrals assigned in this course.

Tabular integration is a useful bookkeeping scheme for performing integration by parts multiple times in a row. The basic technique is to split the integrand into to pieces, iteratively integrate one part and integrate the other, and arrange the results into a table. For example, consider $\int x^2 e^{2x}$. The integrand $x^2 e^{2x}$ can be split into the two parts x^2 and e^{2x}. Now differentiate x^2 until it reaches 0, and meanwhile integrate e^{2x} repeatedly (omitting the $+C$ terms in this case; the only thing that matters is that the derivative of each entry of the second column is equal to the function in the cell above it). The following table results.

$$\begin{tikzcd}
x^2 \arrow[d, dotted, "\frac{d}{dx}"'] & e^{2x} \\
2x \arrow[d, dotted, "\frac{d}{dx}"'] & \frac{1}{2}e^{2x} \arrow[u, dotted, "\frac{d}{dx}"'] \\
2 \arrow[d, dotted, "\frac{d}{dx}"'] & \frac{1}{4}e^{2x} \arrow[u, dotted, "\frac{d}{dx}"'] \\
0 & \frac{1}{8}e^{2x} \arrow[u, dotted, "\frac{d}{dx}"']
\end{tikzcd}$$

I have indicated with arrows the direction in which differentiation is taking place. So the lett column was created by successive differentiation, and the right column by successive integration.

To obtain the indefinite integral from this table, draw slanted lines with alternating signs, take the product of the terms at the end of each arrow, with the sign on the arrow, and sum. The process is shown below.

$$
\begin{tikzcd}
x^2 \arrow[dr, "+"'] & e^{2x} \\
2x \arrow[dr, "-"'] & \frac{1}{2}e^{2x} \\
2 \arrow[dr, "+"'] & \frac{1}{4}e^{2x} \\
0 & \frac{1}{8}e^{2x}
\end{tikzcd}
$$

$$
\begin{aligned}
\int x^2 e^{2x} &= x^2 \cdot \frac{1}{2}e^{2x} - 2x \cdot \frac{1}{4}e^{2x} + 2 \cdot \frac{1}{8}e^{2x} + C \\
&= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C
\end{aligned}
$$

This technique will always work when the left column eventually becomes 0, that is, when the top of the left column is a polynomial. However, with a slight modification, the technique of tabular integration can be useful even when the left column does not eventually become 0. The modification is to add a horizontal line on the bottom row, give it a sign (in the same alternating fashion), and include the integral of the product of these terms (with the sign) in the sum.

For example, here is how you can use this more general type of tabular integration on $\int e^{2x}\sin(3x)dx$.

$$
\begin{tikzcd}
e^{2x} \arrow[dr, "+"'] & \sin(3x) \\
2e^{2x} \arrow[dr, "-"'] & -\frac{1}{3}\cos(3x) \\
4e^{2x} \arrow[r, "+\int"'] & -\frac{1}{9}\sin(3x)
\end{tikzcd}
$$

$$
\begin{aligned}
\int e^{2x}\sin(3)dx &= e^{2x} \cdot \left(-\frac{1}{3}\cos(3x)\right) - 2e^{2x} \cdot \left(-\frac{1}{9}\sin(3x)\right) + \int 4e^{2x} \cdot \left(-\frac{1}{9}\sin(3x)\right) dx \\
&= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\int e^{2x}\sin(3x)dx
\end{aligned}
$$

From here, both indefinite integrals can be moved to one side of the equation, and then computed by solving.

Notice that the reason a horizontal line with an integral is not necessary when the left column becomes 0 is quite simple: the integral is still there, but it is equal to 0.

Notice also that the usual version of integration by parts in general can be shown by a tabular integration with only two rows.

$$
\begin{tikzcd}
u(x) \arrow[dr, "+"'] & v'(x) \\
u'(x) \arrow[r, "-\int"'] & v(x)
\end{tikzcd}
$$

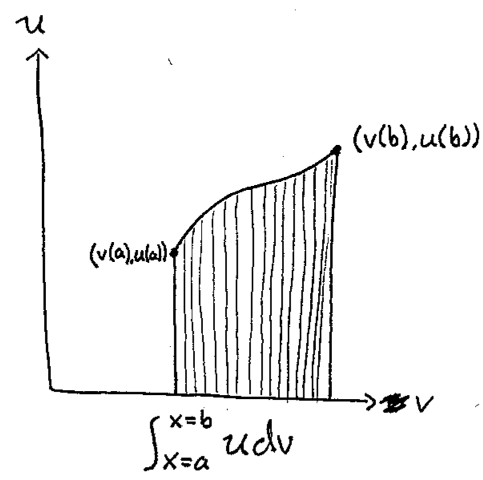
$$\begin{aligned}\int u(x)v'(x)dx &= u(x)\cdot v(x) - \int u'(x)v(x)dx \\ \int udv &= uv - \int vdu\end{aligned}$$

It is not difficult to see how iterating this process produces the general technique of tabular integration as described above.

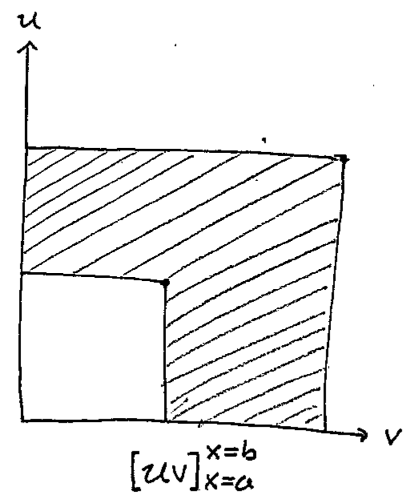
6 Appendix: geometric interpretation

Integration by parts can be viewed as a generalization of the idea that an integral can be computed by slicing either horizontally or vertically. This appendix is meant to explain how this works.

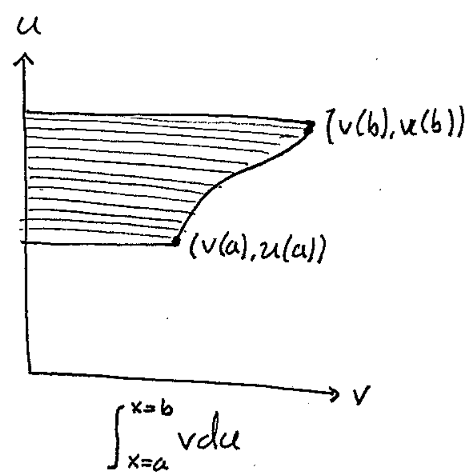
Suppose that $u(x), v(x)$ are functions of x, usually abbreviated u, v, and use as usual the shorthand $du = u'(x)dx$ and $dv = v'(x)dx$. Suppose that we desire to compute the integral $\int_{x=a}^{x=b} udv$. We can visualize the value of this integral by plotting a curve with coordinates $(v(x), u(x)))$ from $x = a$ to $x = b$; the desired integral is the area under this curve, as shown in the picture.



Now, we can attempt to integrate one "one part" of the integral $\int udv$, in effect assuming that u is constant, and guess that the indefinite integral is simply uv. What will be the error of this guess? The guess for the definite integral would be $[uv]_{x=a}^{x=b} = u(b)v(b) - u(a)v(b)$, which can be seen geometrically as the L-shaped regions shown.



The error of this guess is precisely the area to the left of the curve, which can be computed by means of horizontal slicing; it is precisely equal to $\int_{x=a}^{x=b} vdu$.



Therefore the formula for integration by parts of a definite integral is obtained.

$$\int_{x=a}^{x=b} udv = [uv]_{x=a}^{x=b} - \int_{x=a}^{x=b} vdu$$

The geometric effect of this transformation is to convert the effort of evaluating the original integral (sliced vertically) to evaluating a new, horizontally sliced integral, which may be easier. An example of an application which comes to mind from this viewpoint is described in the following appendix.

7 Appendix: integrating inverse functions

The integral of an inverse function can often be more easily computed by "slicing horizontally," that is, transforming an integral that was originally with respect to x to an integral with respect to y. This is the sort of transformation that integration by parts accomplishes, although integration by parts is a much more general technique. The relationship is most easily seen when attempting to integrate inverse functions, such as $\ln x$ or $\tan^{-1} x$, as in the examples above.

In fact, the same sort of technique work for $f^{-1}x$, whenever $f(x)$ is a function whose inverse can be integrated. To see this, suppose that $f(x)$ is a function with known antiderivative $F(x)$. Then we can compute the integral of $f^{-1}(x)$ as follows. Note that I do not actually write down the derivative of $f^{-1}(x)$ because I do not care; in this case it is not actually needed in order to finish evaluating the integral. I also use the letter y rather than u since I really am thinking of this as a y-coordinate in a graph.

$$\begin{aligned}
\int f^{-1}(x)dx & & y = f^{-1}(x),\ dv = dx \\
& & dy = (\cdots),\ v = x \\
&= & xf^{-1}(x) - \int x dy \\
&= & xf^{-1}(x) - \int f(y)dy \\
&= & xf^{-1}(x) - F(y) + C \\
&= & xf^{-1}(x) - F(f^{-1}(x)) + C
\end{aligned}$$

Notice that this commutes a number of integrals that we have previously computed (and not surprisingly, it gives the same answer).

Example 7.1. Let $f(x) = x^2$, so that $f^{-1}(x) = \sqrt{x}$, and $F(x)$ can be taken to be $\frac{1}{3}x^3$. Then $F(f^{-1}(x)) = \frac{1}{3}(\sqrt{x})^3 = \frac{1}{3}x^{3/2}$. Therefore

$$\begin{aligned}
\int \sqrt{x}dx &= x\sqrt{x} - \frac{1}{3}x^{3/2} + C \\
&= \frac{2}{3}x^{3/2} + C
\end{aligned}$$

Example 7.2. Let $f(x) = e^x$. Then we can take $F(x)$ to be e^x as well, and this last result gives:

$$\begin{aligned}
\int \ln x dx &= x \ln x - e^{\ln x} + C \\
&= x \ln x - x + C
\end{aligned}$$

Example 7.3. Let $f(x) = \tan x$. Then we can take $F(x)$ to be $\ln|\sec x|$ (see lecture 30). Therefore $F(f^{-1}(x)) = \ln|\sec(\tan^{-1} x)| = \ln|\sqrt{1+x^2}| = \frac{1}{2}\ln|1+x^2|$. The penultimate step follows because if $\theta = \tan^{-1} x$, then θ is the angle opposite x in a right triangle with legs of length x and 1, hence the hypotenuse is of length $\sqrt{1+x^2}$, and $\sec\theta = \sqrt{1+x^2}$. Therefore we obtain

$$\int \tan^{-1} x = x \tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C$$