Lecture 31: Substitution II

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1 Introduction

Strictly speaking, there's no new material today. We will work through more examples of computing integrals by substitution. The new content consists of several new heuristics, which show slightly more clever or elaborate uses of substitution. I list the examples first, and strongly suggest that you attempt them to the best of your ability before reading the solutions and commentary.

A blanket remark: the main defect of this lecture is that I mainly only show *successful* solution attempts, meaning that I don't include the many false starts and failed attempts that inevitably happen (especially for beginners). This is why it is especially important that you attempt to work many problems yourself, to see both how things can go wrong and how they can go right.

The reference for today is Stewart §5.5.

2 Problem list

We will consider the following integrals. For reference, the answers are provided at the end. Solutions and commentary (which describe the heuristics I mean to illustrate with each problem) are in the middle.

1. $$\int_0^\pi \frac{e^x + \cos x}{e^x + \sin x} dx$$

2. $$\int x\sqrt{x+1}\; dx$$

3. $$\int x^3 (x^2+1)^{21} dx$$

4. $$\int_{\ln 2}^{\ln 3} \frac{e^{2x}}{e^x + 1} dx$$

5. $$\int \frac{1}{x\left((\ln x)^2 + 4\ln x + 4\right)} dx$$

6.

$$\int \frac{1}{4+x^2}dx$$

7.

$$\int \cos^3 x \sin^2 x dx$$

8.

$$\int \cos(\sqrt{x})dx$$

3 Heuristics for substituion

Each subsection illustrates a different heuristic (informal problem solving method) that is often useful for evaluating integrals using substitution. None of these is a silver bullet, and you should not be afraid to try any other substitutions that might occur to you.

3.1 Try to substitute the "innermost function"

This was demonstrated at some length in the previous lecture, so I will not elaborate further here. See in particular these examples: $\int x^2\sqrt{x^3+1}dx$, $\int xe^{x^2}dx$, $\int e^x\cos(e^x)dx$, all of which were discussed in the previous lecture.

3.2 Try substituting the denominator

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Consider this example.

$$\int_0^\pi \frac{e^x+\cos x}{e^x+\sin x}dx$$

A couple substitutions you might try are $u = e^x, u = \sin x, u = \cos x$. None of these will be successful, largely because you will have a hard time expressing $\sin x$ in terms of e^x (for example).

In this case, what does work is substituting for the entire denominator: $u = e^x + \sin x$. Luckily, the numerator comes out to be exactly equal to the derivative of u with respect to x: $du = (e^x + \cos x)dx$. So the integral yields immediately to this substitution.

$$\begin{aligned}
\int_0^\pi \frac{e^x+\cos x}{e^x+\sin x}dx & \qquad u = e^x+\sin x,\ du = (e^x+\cos x)dx \\
&= \int_{1+0}^{e^\pi+0} \frac{du}{u} \\
&= [ln|u|]_1^{e^\pi} \\
&= \ln|e^\pi| - 0 \\
&= \pi
\end{aligned}$$

3.3 Express x in terms of u, and dx in terms of du

Consider the second example: $\int x\sqrt{x+1}\ dx$. In this case, one substitution leaps out: take the "innermost function" $u = x+1$. Since $du = dx$, the integral becomes $\int x\sqrt{u}du$. But what can be done about that x? The answer is that you can just write x in terms of u: $x = u-1$. Then the integral becomes the tractable $\int(u-1)\sqrt{u}$. The full computation is as follows.

$$
\begin{aligned}
\int x\sqrt{x+1}dx & \qquad u = x+1,\ du = dx \\
&= \int (u-1)\sqrt{u}du \\
&= \int (u^{3/2} - u^{1/2})du \\
&= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C
\end{aligned}
$$

Next consider example 3: $\int x^3(x^2+1)^{21}dx$. If you really wanted to, you could do this integral by expanding out this 21st power, but this would frankly ruin your day. So instead try to most suggestive substitution: $u = x^2+1,\ du = 2xdx$. But how can you re-express the integrand? One neat trick is to write du in terms of dx: $du = \frac{1}{2x}dx$. Then the integral becomes

$$\int x^3u^{21}\frac{du}{2x} = \int \frac{1}{2}x^2u^{21}du$$

This still isn't an integral purely in u! But the only remaining vestige of x is that term x^2, which is easy enough to express in terms of u: since $u = x^2+1$, we know that $x^2 = u-1$. So this integral can be evaluated as follows.

$$
\begin{aligned}
\int x^3(x^2+1)^{21}dx & \qquad u = x^2+1,\ du = 2xdx \\
&= \int \frac{1}{2}(u-1)u^{21}du \\
&= \int \left(\frac{1}{2}u^{22} - \frac{1}{2}u^{21}\right) du \\
&= \frac{1}{2\cdot 23}u^{23} + \frac{1}{2\cdot 22}u^{22} + C \\
&= \frac{1}{46}(x^2+1)^{23} - \frac{1}{44}(x^2+1)^{22} + C
\end{aligned}
$$

The next example is similar. In this case, we try substituting the denominator, and must do a little work to re-express the integrand purely in terms of u. As in the previous examples, I end up passing through some steps where the integrand is in a sort of mixed state: partly in x and partly in u. The goal, as usual, is to turn it into something purely in u.

$$
\begin{aligned}
\int_{\ln 2}^{\ln 3} \frac{e^{2x}}{e^x+1}dx & \qquad u = e^x + 1,\ du = e^x dx \\
&= \int_{2+1}^{3+1} \frac{e^{2x}}{u}\frac{du}{e^x} \\
&= \int_3^4 \frac{e^x}{u} du \\
&= \int_3^4 \frac{u-1}{u} du \\
&= \int_3^4 \left(1 - \frac{1}{u}\right) du \\
&= [u - \ln|u|]_3^4 \\
&= (4 - \ln 4) - (3 - \ln 3) \\
&= 1 - \ln 4 + \ln 3
\end{aligned}
$$

3.4 Perform multiple substitutions

Consider the next example. A good substitution to attempt is $u = \ln x$ (innermost function), but it doesn't get us quite far enough to finish.

$$
\begin{aligned}
\int \frac{1}{x\left((\ln x)^2 + 4\ln x + 4\right)} dx & \qquad u = \ln x,\ du = \frac{1}{x} dx \\
&= \int \frac{1}{u^2 + 4u + 4} \frac{dx}{x} \\
&= \int \frac{1}{u^2 + 4u + 4} du
\end{aligned}
$$

This substitution made the integral simpler, but apparently not simple enough. But the new form is simple enough to reveal the next step: factor the bottom into $(u+2)^2$, and perform a second substitution. The traditional letter to use for a second substitution is v (though of course you could use y, Ω, or Ξ for all I care).

$$
\begin{aligned}
\int \frac{1}{u^2 + 4u + 4} du &= \int \frac{1}{(u+2)^2} du \\
& \qquad v = u + 2,\ dv = du \\
&= \int \frac{1}{v^2} dv \\
&= -\frac{1}{v} + C \\
&= -\frac{1}{u+2} + C \\
&= -\frac{1}{\ln x + 2} + C
\end{aligned}
$$

Note. Although it is very helpful practically to performs substitutions several times in a row like this, you could also collapse the two together by just making the substitution $u = \ln x + 2$ from the very beginning. This is great if you are clever enough to think of that, but it's pretty difficult to come up with out of the blue.

3.5 Re-arrange, re-express

The next three examples are somewhat more difficult. They all share a common theme: perform some algebraic manipulations to re-express the integral as something more tractable.

$$\begin{aligned}
\int \frac{1}{4+x^2}dx &= \int \frac{1/4}{1+x^2/4}dx \\
&= \int \frac{1}{4}\frac{1}{1+(x/2)^2}dx \\
u = x/2,\ du = \frac{1}{2}dx & \\
&= \frac{1}{4}\int \frac{2}{1+u^2}du \\
&= \frac{1}{4}\cdot 2\tan^{-1} u + C \\
&= \frac{1}{2}\tan^{-1}(x/2) + C
\end{aligned}$$

Consider the following trigonometric integral.

$$\int \cos^3 x \sin^2 x dx$$

There are two natural choices for substitution: $u = \cos x$ and $u = \sin x$. I recommend that you try $u = \cos x$, and take note of where you get stuck. It is difficult to evaluate the integral that way. But it is possible if you substitute $u = \sin x$ instead.

$$\begin{aligned}
\int \cos^3 x \sin^2 x dx \qquad & u = \sin x,\ du = \cos x dx \\
&= \int \cos^3 x \cdot u^2 \frac{du}{\cos x} \\
&= \int \cos^2 x \cdot u^2 du
\end{aligned}$$

At this stage, the key is the following re-expression step: write $\cos^2 x = 1 - \sin^2 x$, i.e. $1 - u^2$. This will complete the computation.

$$\begin{aligned}
\int \cos^2 x u^2 du &= \int (1-u^2)u^2 du \\
&= \int (u^2 - u^4)du \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\
&= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C
\end{aligned}$$

3.6 Don't always expect to finish

This isn't a heuristic so much as a reminder of a basic fact of life: sometimes best efforts cannot be successful without introducing new methods. For example, the most prudent substitution for the last example simply does not finish off the problem.

$$\begin{aligned}\int \cos(\sqrt{x})dx \qquad & u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}dx \\ &= \int \cos(u) 2\sqrt{x} du \\ &= \int 2u \cos u \ du\end{aligned}$$

In fact, I am not aware of a way to evaluate this integral using the method of substitution alone. In this case, we have successfully transformed the integral into something "simpler" (by eliminating the square root), but our work is not yet finished: we now must evaluate $\int 2u \cos u du$. In fact, there is an effective way to achieve this, which is integration by parts. We'll return to this example next time. Of course, you don't *need* integration by parts in a real sense, since you can also find the antiderivative with a little cleverness, but either way I won't give the answer until the next lecture.

4 Answers to all problems

1. $\displaystyle\int_0^\pi \frac{e^x + \cos x}{e^x + \sin x} dx = \pi.$
2. $\displaystyle\int x\sqrt{x+1}dx = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.$
3. $\displaystyle\int x^3(x^2+1)^{21}dx = \frac{1}{46}(x^2+1)^{23} - \frac{1}{44}(x^2+1)^{22} + C.$
4. $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{2x}}{e^x+1} dx = 1 - \ln 4 + \ln 3.$
5. $\displaystyle\int \frac{1}{x\left((\ln x)^2 + 4\ln x + 4\right)} dx = -\frac{1}{\ln x + 2} + C.$
6. $\displaystyle\int \frac{1}{4+x^2} dx = \frac{1}{2}\tan^{-1}(x/2) + C.$
7. $\displaystyle\int \cos^3 x \sin^2 x dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$
8. $\displaystyle\int \cos(\sqrt{x})dx$ will be finished next time. It is easiest to compute by starting with substation, and finishing with integration by parts.