Lecture 30: Substitution I

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1 Introduction

Every differentiation method has a corresponding integration method. Today we will consider the integration equivalent of the chain rule, which is called "integration by substitution." Next week we'll consider the integration equivalent of the product rule, which is called "integration by parts."

Substitution is used primarily to simplify integrals by collapsing large parts of the integrand into a single (new) variable. Ideally, the result is an integral that can be computed easily by a known antiderivative. We'll consider easier cases in this lecture; in the next we'll examine more complex cases were multiple steps or additional alchemy is needed.

The reference for today is Stewart §5.5.

2 Reversing the chain rule

I'll be begin with an example. By the chain rule, we know that

$$\frac{d}{dx}(\ln x)^2 = 2\ln x \cdot \frac{1}{x}$$

This means, of course, that $(\ln x)^2$ is an antiderivative of $\frac{2\ln x}{x}$. In other words,

$$\int \frac{2\ln x}{x} dx = (\ln x)^2 + C$$

But suppose that we had to go the other way. If you are confronted with $\int \frac{2\ln x}{x} dx$, how could it possibly strike you that this integrand is the derivative of $(\ln x)^2$? The basic idea of substitution is that you can sometimes detect that the integrand has come out of the chain rule.

Remember the statement of the chain rule. Here I've denoted the inner function by the letter u to match a standard convention for integration by substitution, and capital F for the outer function for reasons that will become clear.

$$F(u(x))' = F'(u(x)) \cdot u'(x)$$

Following the usual convention that $f(x)$ is $F'(x)$, this can be written:

$$F(u(x))' = f(u(x)) \cdot u'(x)$$

This equation can be restated in terms of antiderivatives as follows.

$$\int f(u(x)) \cdot u'(x) dx = F(u(x)) + C$$

Another way to write this is the following.

$$\int f(u(x)) \cdot u'(x)dx = \int f(u)du$$

So in order to reverse the chain rule to evaluate an integral, it is necessary to identify a function $u(x)$ such that you see $u'(x)$ in the integrand, and everything else is a function of $u(x)$.

Return to our example: $\int \frac{2\ln x}{x}dx$. In this example, the most complex part of the integral is $\ln t$, and luckily we see its derivative in the integral as well. So in fact we can re-express the integral as follows.

$$\begin{aligned}
\text{let } u(x) &= \ln x \\
\text{so } u'(x) &= \frac{1}{x} \\
\frac{2\ln x}{x} &= 2\ln x \cdot \frac{1}{x} \\
&= 2u(x)u'(x) \\
\Rightarrow \int \frac{2\ln x}{x}dx &= \int 2u(x)u'(x)dx \\
&= \int 2udu \\
&= u^2 + C \\
&= (\ln x)^2 + C
\end{aligned}$$

So the steps are: identify a function $u(x)$, rewrite the integrand as a product of a function of $u(x)$ $(f(u(x)))$ times $u'(x)$, integrate $f(u)$ with respect to u, and finally replace u by its definition again. These steps will become much more clear and be easier to recall in examples.

First, I'll redo this example with less explanation, to show the standard shorthand used for this kind of work.

3 The usual shorthand

The standard way to compute our example in the previous section is the following.

$$\begin{aligned}
\int \frac{2\ln x}{x}dx & \qquad u = \ln x,\ du = \frac{1}{x}dx \\
&= \int 2\ln x \cdot \frac{1}{x}dx \\
&= \int 2udu \\
&= u^2 + C \\
&= (\ln x)^2 + C
\end{aligned}$$

In this shorthand:

- The chosen function $u(x)$ is denoted simply by u.
- The expression $du = \frac{1}{x}dx$ really just asserts that $\frac{du}{dx} = \frac{1}{x}$. It is a way to record the derivative $u'(x)$.
- By replacing $u'(x)dx$ by du and everything else in sight as some function of u, we obtain an integral which is now written entirely as an integral in u rather than in x.

This method is called substitution because you substitute one variable (x) for another (traditionally u, but sometimes v; it can be anything you like). After this substitution, the integral is transformed into a (hopefully) easier integral with a different variable.

4 Definite integrals by substitution

So far we've only discussed indefinite integrals (antiderivatives) by substitution. To do definite integrals by substitution is almost identical; the only difference is that when you transform from the integral in x to the integral in u, you should also modify the endpoints by applying the function $u(x)$ to them. For example:

$$\begin{aligned}
\int_1^2 \frac{2\ln x}{x}dx & \qquad u = \ln x,\ du = \frac{1}{x}dx \\
&= \int_{\ln 1}^{\ln 2} 2udu \\
&= \int_0^{\ln 2} 2udu \\
&= \left[u^2\right]_0^{\ln 2} \\
&= (\ln 2)^2
\end{aligned}$$

The only line worth remarking on here is the second: notice that once I write the integral in terms of u (you can tell because I've written a du), then I must replace the endpoints 1 and 2 by $\ln 1$ and $\ln 2$.

There is a second option, which is the actually finish taking the antiderivative (as a function in x) and substituting the original limits. In this case, you should clearly indicate that the endpoints are values of x, not values of u. Here's a way to write it that would avoid confusion.

$$\begin{aligned}
\int_1^2 \frac{2\ln x}{x}dx & \qquad u = \ln x,\ du = \frac{1}{x}dx \\
&= \int_{x=1}^{x=2} 2udu \\
&= \left[u^2\right]_{x=1}^{x=2} \\
&= \left[(\ln x)^2\right]_{x=1}^{x=2} \\
&= (\ln 2)^2 - (\ln 1)^2 \\
&= (\ln 2)^2
\end{aligned}$$

Here I have clearly marked the values 1 and 2 as values of x after making the transition to an integral in u. This is very important, so that you don't accidentally substitute $u = 1$ and $u = 2$ at the end, which would give an incorrect answer.

I strongly suggest that you stick to the first method. You're going to end up applying the function $u(x)$ to the endpoints no matter what, so you might as well do it sooner rather than later. The antiderivative will almost certainly look nicer in terms of u than in terms of x anyway.

5 Examples

The only way to learn how to do integrals by substation is practice and looking at examples. It takes some good instincts to identify the right u to use in substitution, so you should look at many examples to get a feel for what tends to work. In the examples below I have tried to highlight how I know to make certain substitutions in the commentary.

The homework has many more examples, and the textbook has many worked examples as well.

5.1 Linear substitutions

These are substitutions where you take u to be a linear function of x. These are fairly common; you can use them when you see something like $-x$ or $2x+1$ and you'd know what to do if it just had the simple human decency to be an x instead.

Example 5.1.

$$\begin{aligned}
\int e^{-2x}dx & \qquad u=-2x,\ du=-2dx \\
&= \int e^{-2x}\cdot(-\frac{1}{2})(-2)dx \quad \text{(introduce the factor of (-2) to create } du\text{)} \\
&= \int e^{u}\cdot(-\frac{1}{2})du \\
&= -\frac{1}{2}e^{u} \\
&= -\frac{1}{2}e^{-2x}
\end{aligned}$$

In this example, I would know what to do if that $-2x$ where just an x (or a u), so the substitution immediately makes the integral tractable.

Example 5.2.

$$\begin{aligned}
\int_2^9 \frac{1}{\sqrt{x+7}}dx & \qquad u=x+7,\ du=dx \\
&= \int_{2+7}^{9+7}\frac{1}{\sqrt{u}}du \\
&= \left[2\sqrt{u}\right]_9^{16} \\
&= 2\sqrt{16}-2\sqrt{9} \\
&= 8-6 \\
&= 2
\end{aligned}$$

Example 5.3.

$$\begin{aligned}
\int \cos(5x+7)dx & \qquad u=5x+7,\ du=5dx \\
&= \int \cos u\cdot\frac{1}{5}du \\
&= \frac{1}{5}\sin u+C \\
&= \frac{1}{5}\sin(5x+7)+C
\end{aligned}$$

5.2 Polynomial substitutions

Example 5.4.

$$
\begin{aligned}
\int_0^4 xe^{x^2}dx & \qquad u = x^2,\ du = 2xdx \\
&= \int_0^4 e^{x^2} \cdot \frac{1}{2} \cdot 2xdx \\
&= \int_0^{16} \frac{1}{2}e^u du \\
&= \left[\frac{1}{2}e^u\right]_0^{16} \\
&= \frac{1}{2}e^{16} - \frac{1}{2}
\end{aligned}
$$

In this case, you might guess that $u = x^2$ is a good substitution because it is the "innermost function," up in the exponent of e^{x^2}. This is a good rule of thumb, which is also shown in the next example.

Example 5.5.

$$
\begin{aligned}
\int x^2\sqrt{x^3+1}\ dx & \qquad u = x^3+1,\ du = 3x^2dx \\
&= \int \sqrt{u}x^2dx \\
&= \int \sqrt{u} \cdot \frac{1}{3}du \\
&= \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{9}u^{3/2} + C \\
&= \frac{2}{9}(x^3+1)^{3/2} + C
\end{aligned}
$$

Example 5.6.

$$
\begin{aligned}
\int_0^2 3x\sqrt{x^2+1}\ dx & \qquad u = x^2+1,\ du = 2xdx \\
&= \int_1^5 \sqrt{u} \cdot \frac{3}{2}du \\
&= \left[\frac{3}{2} \cdot \frac{2}{3} \cdot u^{3/2}\right]_1^5 \\
&= \left[u^{3/2}\right]_1^5 \\
&= 5^{3/2} - 1
\end{aligned}
$$

5.3 Trigonometric substitutions

Example 5.7.

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \sin^3 x \cos x \; dx \qquad & u = \sin x,\; du = \cos x dx \\ &= \int_{1/2}^{\sqrt{3}/2} u^3 du \\ &= \left[\frac{1}{4}u^4\right]_{1/2}^{\sqrt{3}/2} \\ &= \frac{1}{4}\cdot\frac{9}{16} - \frac{1}{4}\cdot\frac{1}{16} \\ &= \frac{8}{64} \\ &= \frac{1}{8}\end{aligned}$$

Example 5.8.

$$\begin{aligned}\int \frac{\cos x}{\sin^2 x} \; dx \qquad & u = \sin x,\; du = \cos x dx \\ &= \int \frac{1}{u^2} du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\sin x} + C\end{aligned}$$

Example 5.9.

$$\begin{aligned}\int_0^{\pi/4} \tan x \; dx \qquad & u = \cos x,\; du = -\sin x dx \\ &= \int_0^{\pi/4} \frac{\sin x}{\cos x} dx \\ &= \int_1^{\sqrt{2}/2} \frac{-1}{u} du \\ &= [-\ln |u|]_1^{\sqrt{2}/2} \\ &= -\ln \frac{\sqrt{2}}{2} \\ &= \ln \frac{2}{\sqrt{2}} = \ln \sqrt{2} \\ &= \frac{1}{2}\ln 2\end{aligned}$$

5.4 Other examples

Example 5.10.

$$\begin{aligned}\int e^x \cos(e^x)dx & \qquad u = e^x,\ du = e^x dx \\ &= \int \cos(e^x)e^x dx \\ &= \int cos u du \\ &= \sin u + C \\ &= \sin(e^x) + C\end{aligned}$$

Example 5.11.

$$\begin{aligned}\int \frac{(\ln x)^7}{x}dx & \qquad u = \ln x,\ du = \frac{1}{x}dx \\ &= \int u^7 du \\ &= \frac{1}{8}u^8 + C \\ &= \frac{1}{8}(\ln x)^8 + C\end{aligned}$$