Lecture 24: Indeterminate forms

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1 Introduction

Last time, we saw one method for dealing with limits of functions $\frac{f(x)}{g(x)}$, where both $f(x)$ and $g(x)$ have limits of ∞ or 0; such limits are said to be in the indeterminate forms $\frac{\infty}{\infty}$ or $\frac{0}{0}$. Today we will examine several other similar situation, where a function is built up from two other functions, but the limits of these two functions is not sufficient information to determine the overall limit. In most such cases, the necessary idea is to somehow reformulate the limit in a form that previous techniques can approach.

The reference for today is Stewart §4.5.

2 List of indeterminate forms

The following expressions are all called indeterminate forms.

$$\begin{gathered} \frac{0}{0} \\ \frac{\infty}{\infty} \\ 0 \cdot \infty \\ \infty^0 \\ 0^0 \\ 1^\infty \\ \infty - \infty \end{gathered}$$

Remember that ∞ is not a real, honest number, but a shorthand for a limiting process. Here are more precise statements of what it means that each of these forms are indeterminate:

- $\frac{0}{0}$ is indeterminate, because knowing that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ is not enough information to determine $\lim_{x\to c} \dfrac{f(x)}{g(x)}$.
- $\frac{\infty}{\infty}$ is indeterminate, because knowing that $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$ is not enough information to determine $\lim_{x\to c} \dfrac{f(x)}{g(x)}$.
- $0 \cdot \infty$ is indeterminate, because knowing that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$ is not enough information to determine $\lim_{x\to c} f(x) \cdot g(x)$.
- ∞^0 is indeterminate, because knowing that $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = 0$ is not enough information to determine $\lim_{x\to c} f(x)^{g(x)}$.

- 0^0 is indeterminate, because knowing that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ is not enough information to determine $\lim_{x\to c} f(x)^{g(x)}$.
- 1^∞ is indeterminate, because knowing that $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = \infty$ is not enough information to determine $\lim_{x\to c} f(x)^{g(x)}$.
- $\infty - \infty$ is indeterminate, because knowing that $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$ is not enough information to determine $\lim_{x\to c}(f(x) - g(x))$.

You've found some examples in your homework to show that these forms are all indeterminate.

In contast, forms like $\frac{0}{\infty}$, $\infty + \infty$ and 0^∞ are determinate, because if they arise in a limit, the answer is unambiguous (in these cases, these forms resolve to $0, \infty, 0$, respectively).

Keeping straight which forms resolve unambiguously, and which are indeterminate, can be confusing. Here is a good rule of thumb: **a form is indeterminate if it is sensitive to small errors.** Here's what I mean:

- Consider $\frac{0}{0}$. Think of both of these zeros as being imprecise measurements. So it could be that one zero is actually 0.0001, while the other is actually 0.0000001, but your measuring device is only accurate enough to measure them both as zero. Then the ratio could be $\frac{0.0001}{0.0000001} = 1000$, but it could also be $\frac{0.0000001}{0.0001} = 0.001$. So this arithmetic is sensitive to errors in the measurements, so it is indeterminate.
- Consider $\frac{\infty}{\infty}$. Think of ∞ as just meaning "off the chart;" it is a number too large for you to measure. So one ∞ could stand for 1000, while the other stands for 1000000. The ratio of these could be 1000, but it could also be 0.001. So the arithmetic is sensitive to error about just how big these two "off the charts" numbers are.
- Consider $\frac{0}{\infty}$. Here 0 is just some small number (possibly with a bit of error), and ∞ is some huge number (off the charts of your measuring device). But you can see that even if you don't know just how far off the charts the ∞ is, or just how close to 0 the 0 is, the quotient will nonetheless be an immeasurably tiny number. So this form is perfectly well-defined: it is definitely 0.

If you apply this criterion carefully, you will be able to tell whether a particular form determines the limit uniquely or not. For example, consider the most confusing indeterminate form of the bunch: 1^∞. Then all you really know about the ∞ is that it's too large to measure; lets say it is at least 1000000. And all you know about the 1 is that it is within 0.0001 of 1. This isn't good enough to have even an approximate sense for the result of exponentiation: it might actually be $1.0001^{1000000} \approx 10^{43}$, but it could also be $0.999^{1000000} \approx 10^{-43}$. This arithmetic is extremely sensitive to even small errors in the measurement of "1" and "∞," so it is indeterminate.

The point is that if you obtain an indeterminate form when you evaluate the limits of two parts of a function separately, you will need to do some more analysis to find the limit. We saw one tool last time that often resolves $\frac{0}{0}$ or $\frac{\infty}{\infty}$, called L'Hôpital's rule. We will now examine some examples of how to resolve the other types. This will often involve converting these other types into a from where L'Hôptial's rule applies, but not always; in some cases we will want to draw on more elementary techniques.

3 Resolving $\infty \cdot 0$

The typical way to resolve $\infty \cdot 0$ is to convert it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by either taking the reciprocal of the ∞ and putting it in the denominator, or putting the reciprocal of the 0 and putting it in the denominator. Here is an example of both of these.

Example 3.1. Consider $\lim_{x\to\infty}\left(x\cdot\sin\left(\frac{2}{x}\right)\right)$. This has the form $\infty\cdot 0$ if you try to evaluate directly. But if you flip the x to the denominator

$$\lim_{x\to\infty}\left(\frac{\sin\left(\frac{2}{x}\right)}{1/x}\right)$$

then this limit has the form $\frac{0}{0}$. We have simply taken the ∞, and transformed it into a 0 in the denominator. This limit can be done with L'Hôpital's rule.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{\sin\left(\frac{2}{x}\right)}{1/x}\right) &= \lim_{x\to\infty}\frac{\cos\left(\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)}{-1/x^2}\\
&= \lim_{x\to\infty} 2\cos\left(\frac{2}{x}\right)\\
&= 2\cos(2/\infty)\\
&= 2\cos(0)\\
&= 2
\end{aligned}$$

So in this case, we could evaluate the limit by flipping it to $\frac{0}{0}$ and using L'Hôpital's rule.

Example 3.2. In this example we'll change $\infty\cdot 0$ to $\frac{\infty}{\infty}$. Consider $\lim_{x\to\infty}(xe^{-x})$. This has the form $\infty\cdot 0$. Move the 0 to an ∞ in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} xe^{-x} &= \lim_{x\to\infty}\frac{x}{e^x}\\
&= \lim_{x\to\infty}\frac{1}{e^x}\\
&= \frac{1}{\infty}\\
&= 0
\end{aligned}$$

It is not always easy to tell whether you should convert $\infty\cdot 0$ to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. You may need to try both options, and see which one makes the problem simpler to solve. As always, there are no hard fast rules; you should just experiment with many examples and try to get a sense for what is most effective in various situations.

4 Resolving 1^∞

In the case of 1^∞, the usual tack is to take the logarithm of the expression in question. This results in a new expression, with will have a limit of the form $\infty\cdot 0$. Resolve this indeterminacy using ideas from the previous section, and then raise e to the result. As a first example, consider the following.

$$\lim_{h\to 0}(1+2h)^{1/h}$$

This can be rewritten as following, by taking the logarithm of the expression.

$$\begin{aligned}
\lim_{h\to 0}(1+2h)^{1/h} &= \lim_{h\to 0} e^{\ln\left((1+2h)^{1/h}\right)}\\
&= e^{\lim_{h\to 0}\ln\left((1+2h)^{1/h}\right)}
\end{aligned}$$

Now evaluate the limit in the exponential separately.

$$\begin{aligned}
\lim_{h\to 0} \ln\left((1+2h)^{1/h}\right) &= \lim_{h\to 0} \frac{1}{h}\ln(1+2h) \quad \text{(has form } \infty\cdot 0\text{)} \\
&= \lim_{h\to 0} \frac{\ln(1+2h)}{h} \quad \text{(Convert to } \tfrac{0}{0}\text{)} \\
&= \lim_{h\to 0} \frac{2/(1+2h)}{1} \quad \text{(L'Hôpital)} \\
&= 2
\end{aligned}$$

Therefore the original limit is:

$$\lim_{h\to 0}(1+2h)^{1/h} = e^2.$$

The reason the logarithm is a useful tool here is precisely that it transforms exponentiation into multiplication. This is one of the main reasons that logarithms became centrally important in mathematics: they convert complicated operations into simpler operations, and therefore make many computations much easier. We've already seen this feature once in this course, in the context of logarithmic differentiation.

5 Resolving 0^0 and ∞^0

For both 0^0 and ∞^0, the usual technique is the same as for 1^∞: evaluate instead the limit of the logarithm of the expression, and then exponentiate the result. Here are a couple examples.

Example 5.1.

$$\begin{aligned}
\lim_{x\to\infty} x^{1/x} &= e^{\lim_{x\to\infty}\frac{1}{x}\ln x} \\
\lim_{x\to\infty} \frac{1}{x}\ln x &= \lim_{x\to\infty} \frac{\ln x}{x} \quad \text{(form } \tfrac{\infty}{\infty}\text{)} \\
&= \lim_{x\to\infty} \frac{1/x}{1} \quad \text{(L'Hôpital)} \\
&= 0 \\
\Rightarrow \lim_{x\to\infty} x^{1/x} &= e^0 = 1
\end{aligned}$$

Example 5.2.

$$\begin{aligned}
\lim_{x\to 0^+} x^x &= e^{\lim_{x\to 0^+} x\ln x} \\
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \text{(form } \tfrac{\infty}{\infty}\text{)} \\
&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \quad \text{(L'Hôpital)} \\
&= \lim_{x\to 0^+}(-x) \\
&= 0 \\
\Rightarrow \lim_{x\to 0^+} x^x &= e^0 \\
&= 1
\end{aligned}$$

6 Resolving $\infty - \infty$

The typical approach to resolving $\infty - \infty$ is to re-express the expression as a fraction, by finding some common denominator. Typically this fraction will also be in an indeterminate form, which will need to be dealt with. Here are two examples.

Example 6.1. Consider $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$. This can be converted to a fraction by the algebraic trick of "multiplying by the conjugate:"

$$\begin{aligned}\sqrt{x^2+x}-x &= \left(\sqrt{x^2+x}-x\right)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} \\ &= \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} \\ &= \frac{x}{\sqrt{x^2+x}+x}\end{aligned}$$

From here, if you try to evaluate the limit $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}$, you will see that it is in the form $\frac{\infty}{\infty}$. So you could apply L'Hôpital's rule at this stage (and this is one valid way to compute the limit). In this case, it is somewhat easier to evaluate this limit by elementary means, however: divide both the numerator and the denominator by x and substitute directly.

$$\begin{aligned}\frac{x}{\sqrt{x^2+x}+x} &= \frac{x}{\sqrt{x^2+x}+x}\cdot\frac{1/x}{1/x} \\ &= \frac{1}{\frac{1}{x}\sqrt{x^2+x}+1} \\ &= \frac{1}{\sqrt{1+1/x}+1} \\ \Rightarrow \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x} &= \frac{1}{\sqrt{1+0}+1} \\ &= \frac{1}{2}\end{aligned}$$

Therefore $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\frac{1}{2}$ as well.

Note. This computation **did not require L'Hôpital's rule** at all; we used only methods from earlier in the course. In fact, this problem was originally proposed as a problem on last week's midterm exam, but was removed for length reasons.

Here is a second example, where the conversion to a fraction is more transparent, but the ensuing limit computation requires a bit more work.

Example 6.2. Consider the limit $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$. This is in the form $\infty-\infty$ if x comes from the right, or $-\infty+\infty$ if x comes from the left. Either way, it must be reformulated. In this case, just find a common denominator and do the fraction arithmetic.

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\cdot\sin x}$$

Now this is in the form $\frac{0}{0}$. We can attack it with l'Hôpital's rule.

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin x}{x\cdot\sin x} \quad (\text{form } \tfrac{0}{0}) \\
&= \lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x} \quad (\text{L'Hôpital; still has form } \tfrac{0}{0}) \\
&= \lim_{x\to 0}\frac{\sin x}{-x\sin x+2\cos x} \quad (\text{L'Hôpital again}) \\
&= \frac{0}{-0+2} \quad (\text{direct substitution}) \\
&= 0
\end{aligned}
$$

So this limit is equal to 0.