Lecture 23: L'Hôpital's rule

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Finite limits are all alike; every infinite limit approaches infinity after its own fashion.

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1 Introduction

Not all infinities are the same, and not all zeros are the same. This is the basic idea behind a fact called *l'Hôpital's rule*. This rule is a common tool for evaluating limits. The rule addresses the very common situation where one wants the limit of a ratio of two functions, which are both approaching ∞, or both approaching zero.

Like any other tool, l'Hôpital's rule is no panacea. It must often be used in conjunction with other techniques. On the other hand, it is remarkably versatile. In the next lecture we'll see ways that the rule can be used on a broader class of limits by appropriate modifications.

The reference for today is §4.5.

2 First examples

Consider the following limit.

$$\lim_{x\to\infty} \frac{\ln x}{x}$$

If you try to substitute $x = \infty$ directly into the function $\frac{\ln x}{x}$, you will obtain $\frac{\infty}{\infty}$, which is "indeterminate." For us, this means that we don't have enough information to determine the limit – it's not enough to know that both the numerator and the denominator go to infinity. It is tempting to just write $\frac{\infty}{\infty} = 1$, but this is not legitimate – the problem is that these two symbols ∞ may look alike, but they came about in rather different ways. They were approached at *different speeds*.

The basic idea behind l'Hôpital's rule is that when you look at a ratio of two things going to infinity, what matters is *how fast* they are both going to infinity. So different infinities are distinguished by how fast they are approached.

In this case: $\ln x$ approaches ∞ *much more slowly* than x does. Indeed, if you multiply x be 1000, then $\ln x$ only increases by $\ln 1000$, which is about 7. So it should not surprise you to learn that this limit is equal to 0. I will skip a formal proof.

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

One way to be a little more numerically precise about the fact that $\ln x$ grows so much slower than x is to look at the exact construct that measures rate of growth: the derivative. Indeed, $\frac{d}{dx}\ln x = \frac{1}{x}$, while $\frac{d}{dx}x = 1$. One growth rate goes to 0, while the other growth rate holds steady. The basic fact (which I will state precisely in the next section) is that in cases where both the numerator and denominator are going to infinity, it is enough to compute the limit of the ratio of derivatives instead. In this case:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x}{x} &= \lim_{x\to\infty}\frac{1/x}{1}\\ &= 0\end{aligned}$$

Consider a second, somewhat different sort of example.

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

This limit is hard to compute for a different reason: both the numerator and the denominator are going to 0. But the idea is exactly the same: study *how fast* the numerator and the denominator are going to 0. It will turn out that in this case as well, a simple way to evaluate this limit is to instead compute the limit of the ratio of the derivatives. I'll state the precise rule that permits this in the next section.

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1}{x} &= \lim_{x\to 0}\frac{(e^x+1)'}{(x)'}\\ &= \lim_{x\to 0}\frac{e^x}{1}\\ &= 1\end{aligned}$$

Aside: the example above it really pretty backwards, since you may notice that this limit is nothing but the *definition* of the derivative of e^x at $x=0$. So to use knowledge of the derivative function to compute this limit is kind of like writing out a 12×12 multiplication table by hand so that you can look up $2\cdot 3$ on it. Still, it's an easy illustration of why l'Hôpital's rule is plausible, and actually hints at one way to prove it.

3 Statement of the rule, and warnings

The following is the main fact of this lecture. I will not give a proof of it, so you will need to memorize it and use it as a black box.

Theorem 3.1 (L'Hôpital's rule)**.** *Suppose that* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ *or* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$*. Then:*

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists.

In this statement:

- The symbol c can stand for an honest real number, and also for ∞ or $-\infty$.
- The same statement holds if you replace all the limits $\lim_{x\to c}$ are replaced with one-sided limits $\lim_{x\to c^+}$ or $\lim_{x\to c^-}$.

This rule is named after Guillaume de l'Hôpital, who wrote the first systematic textbook on differential calculus (called *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes*; roughly "Analysis of the infinitely small for knowledge about curves") in 1696, in which this rule appeared. The rule was not discovered by l'Hôpital himself, but probably by John Bernoulli (with whom l'Hôpital corresponded extensively). There is an urban legend among mathematicians that l'Hôpital was a rich landowner who paid the Bernoulli brothers for their mathematical work and published it under his own name, but this appears to be a exaggeration. Indeed, l'Hôpital specifically thanks the Bernoulli brothers for their help in the book, and he certainly didn't give the rule his own name himself.

Warnings:

1. Make sure that the numerator and denominator actually do both go to 0 (or to infinity) before applying l'Hôpital's rule.

2. The last part of this statement is sometimes important: if the limit $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ does not exist, that does **not** necessarily mean that the limit $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist. For example, think about the situation $f(x) = (x - \cos x)$ and $g(x) = x$ (this is on your homework). However, if the limit of $f'(x)/g'(x)$ exists, that does guarantee that the limit of $f(x)/g(x)$ exists. In short: if you compute a limit successfully with the rule, great! If not, it's possible that you need to try something else.

Despite these warnings, this rule is actually quite versatile. We'll see in the next lecture some ways it can be used for every broader sorts of situations.

4 Examples

I will show how to use l'Hôpital's rule to solve the following limits. You should of course (as usual) try to evaluate them all yourself before reading the solutions.

1. $\lim_{x\to\infty}\frac{\ln x}{\sqrt[10]{x}}$
2. $\lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x}$
3. $\lim_{x\to 0}\frac{e^x + e^{-x}}{\cos x}$
4. $\lim_{x\to 0}\frac{\sin x}{\tan^{-1} x}$
5. $\lim_{x\to 1}\frac{\cos(\frac{\pi}{2}x)}{\sqrt{x}-1}$
6. $\lim_{x\to\infty}\frac{e^x}{x^4}$
7. $\lim_{x\to\infty}\frac{\ln(\ln x)}{\sqrt{x}}$
8. $\lim_{x\to 0}\frac{x^3}{x - \tan^{-1} x}$
9. $\lim_{x\to\pi}\frac{\sin x + x - \pi}{(x-\pi)^3}$
10. $\lim_{x\to 0}\frac{e^x - 1 - x}{x^2}$

Remember, as always, that there is no clear-cut procedure for any of these problems; you will have to learn from experience when different tools are most useful. I summarize some useful heuristics in the next section, but you should not regard these as universal laws.

Solutions.

1. Both the numerator and denominator in $\lim_{x\to\infty} \frac{\ln x}{\sqrt[10]{x}}$ go to ∞ as $x \to \infty$. So we can differentiate both and take the limit.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln x}{\sqrt[10]{x}} &= \lim_{x\to\infty} \frac{1/x}{\frac{1}{10}x^{-9/10}} \\
&= \lim_{x\to\infty} \frac{10}{x^{1/10}} \\
&= \frac{10}{\infty} \\
&= 0
\end{aligned}$$

So this limit is $\boxed{0}$. Notice in the second line how useful it was to simplify the expression judiciously before proceeding: by moving all of the x factors to the denominator, it was possible to evaluate the limit easily.

Note also in this example that this shows $\ln x$ grows much more slowly than even the tenth root of x. Indeed, one fact that we will see over and over is that $\ln x$ is a truly slow function.

2. Both the numerator and the denominator of $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$ go to 0 as $x \to 0$, so we can differentiate them both and proceed.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} &= \lim_{x\to 0} \frac{e^x - (-e^{-x})}{\cos x} \\
&= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} \\
&= 2
\end{aligned}$$

So this limit equals $\boxed{2}$.

3. The limit $\lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$ can be evaluated by direct substitution (in fact, we already did it at the end of the last problem): it is $\boxed{2}$.

Notice that you should not just blindly apply l'Hôpital's rule to any limit. You must check first whether it really is giving an indeterminate form $0/0$ or ∞/∞.

4. Both the numerator and the denominator of $\lim_{x\to 0} \frac{\sin x}{\tan^{-1} x}$ go to 0. So differentiate them both.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{\tan^{-1} x} &= \lim_{x\to 0} \frac{\cos x}{\frac{1}{1+x^2}} \\
&= \lim_{x\to 0} \cos x(1+x^2) \\
&= 1 \cdot (1+0) \\
&= 1
\end{aligned}$$

So this limit is $\boxed{1}$.

5. The limit $\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{\sqrt{x}-1}$ is in the indeterminate form $0/0$, so differentiate the numerator and the denominator.

$$
\begin{aligned}
\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{\sqrt{x}-1} &= \lim_{x\to 1} \frac{-\sin(\frac{\pi}{2}x)\cdot\frac{\pi}{2}}{\frac{1}{2\sqrt{x}}} \\
&= \lim_{x\to 1} \left(-\sin(\frac{\pi}{2}x)\cdot\frac{\pi}{2}\cdot 2\sqrt{x}\right) \\
&= -1\cdot\frac{\pi}{2}\cdot 2\sqrt{1} \\
&= -\pi
\end{aligned}
$$

So the limit is $\boxed{-\pi}$.

6. The limit $\lim_{x\to\infty} \frac{e^x}{x^4}$ gives the indeterminate form ∞/∞. In this case, you will actually need to apply l'Hôpital several times in a row; each step will make the limit you are trying to evaluate very slightly simpler, until eventually you reach a limit that can be evaluated by direct substitution.

$$
\begin{aligned}
\lim_{x\to\infty} \frac{e^x}{x^4} &= \lim_{x\to\infty} \frac{e^x}{4x^3} \\
&= \lim_{x\to\infty} \frac{e^x}{4\cdot 3x^2} \\
&= \lim_{x\to\infty} \frac{e^x}{4\cdot 3\cdot 2x} \\
&= \lim_{x\to\infty} \frac{e^x}{4\cdot 3\cdot 2} \\
&= \frac{\infty}{4\cdot 3\cdot 2} \\
&= \infty
\end{aligned}
$$

Note that in each line of this argument, the numerator and the denominator both were functions that go to ∞ as $x\to\infty$. It is very important that you always check to make sure this is so before going forward with another differentiation.

So this limit is $\boxed{\infty}$. In works: e^x grows much faster than x^4. In fact, the argument above should convince you that e^x also grows faster than x^5, x^6 or indeed x^{1000}, since in all cases l'Hôpital's rule can simply be applied many times in a row. If you do not see why, try this out for yourself, by evaluating the limits $\lim_{x\to\infty} \frac{e^x}{x^5}$ and $\lim_{x\to\infty} \frac{e^x}{x^6}$; the pattern should become clear.

7. Both the numerator and denominator go to ∞, so we can apply l'Hôpital's rule. The tricky part of this example is computing the derivative of the numerator (with a careful use of the chain rule), and noticing that an appropriate simplification makes the problem easy after that point. Here is the argument.

$$
\begin{aligned}
\lim_{x\to\infty} \frac{\ln(\ln x)}{\sqrt{x}} &= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{d}{dx}\ln x}{\frac{1}{2\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{2\sqrt{x}}{x\ln x} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{x}\cdot\ln x} \\
&= \frac{1}{\infty} \\
&= 0
\end{aligned}
$$

In the fourth line, the key insight for the simplification was that $\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$.

8. Both the top and bottom go to 0 in this case. Therefore:

$$
\lim_{x\to 0} \frac{x^3}{x-\tan^{-1}x} = \lim_{x\to 0} \frac{3x^2}{1-\frac{1}{1+x^2}}
$$

At this stage, the top and bottom still go to 0, and you could differentiate a second time. However, in this case it's quicker just to begin simplifying the expression now, to put it into a tractable form.

$$
\begin{aligned}
\lim_{x\to 0} \frac{x^3}{x-\tan^{-1}x} &= \lim_{x\to 0} \frac{3x^2}{1-\frac{1}{1+x^2}} \\
&= \lim_{x\to 0} \frac{3x^2}{\frac{(1+x^2)-1}{1+x^2}} \\
&= \lim_{x\to 0} \frac{3x^2(1+x^2)}{x^2} \\
&= \lim_{x\to 0} 3(1+x^2) \\
&= 3
\end{aligned}
$$

9. This example can be solved by applying l'Hôpital's rule three times in a row. In each case, the numerator and denominator both go to 0.

$$
\begin{aligned}
\lim_{x\to\pi} \frac{\sin x + x - \pi}{(x-\pi)^3} &= \lim_{x\to\pi} \frac{\cos x + 1}{3(x-\pi)^2} \\
&= \lim_{x\to\pi} \frac{-\sin x}{3\cdot 2(x-\pi)} \\
&= \lim_{x\to\pi} \frac{-\cos x}{3\cdot 2} \\
&= \frac{-(-1)}{6} \\
&= 1/6
\end{aligned}
$$

10. This example can be solved by two applications of l'Hôpital's rule.

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{e^x - 1}{2x} \\ &= \lim_{x\to 0} \frac{e^x}{2} \\ &= \frac{1}{2} \end{aligned}$$

5 Some heuristics

- Do not be afraid to apply l'Hôpital several times in a single problem to simplify the limit repeatedly.
- Before applying l'Hôpital (or any other procedure), it is generally smart to simplify the expression judiciously, to avoid trying to differentiate more complicated functions than you need.
- If you are trying to choose between attempting to use l'Hôpital's rule and some other technique (such as the methods we've studied so far), a good rule of thumb is to attempt whichever method looks easier to carry out. If the numerator or denominator is a rather messy function, it may not be smart to try to differentiate it, for example.
- As in any problem, do not be afraid to try different methods. Some may work, some may fail, so try until you find one that works.