Lecture 20: Further graphing

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1 Introduction

This lecture does not introduce any new material. We revisit the techniques from lecture 12, which give ways to determine the rough qualitative and long-term behavior of functions using some basic calculations with calculus. We previously saw that useful steps in making such a sketch are: determining and classifying singularities; finding local extrema; determining where the function increases/decreases; finding inflection points; determining where the function is concave up/down.

To this list we add two further steps, using recent methods: evaluate infinite limits and limtis at infinity to classify the vertical and horizontal asymptotes. We also briefly discuss a third type of asymptote, called a slant asymptote.

The reference for today is Stewart §4.3.

2 A warm-up question

Here is a question, which I began class with, to get students thinking a bit about what concavity means. (The warmup in class was somewhat lengthier but was mainly review, so I have not included it all here)

Question. Can you think of a function that is increasing and concave up on $(0, \infty)$, and has limit 1 as $x \to \infty$?

The answer to this question turns out to be: no such function is possible. Try to picture what "increasing and concave up" means and you should see why.

Here is a more detailed explanation for why this is impossible: because the function should be increasing, it must have a tangent line with positive slope. In particular, this tangent line will eventually cross over the line $y = 1$ that is meant to be an asymptote. But if the function is concave up, then it must lie *above* its tangent line. But this would mean that it eventually lies above $y = 1$, and will only keep increasing from there! So no such function is possible.

Try to draw a picture to understand this situation. It is a good exercise in thinking about what exactly "concavity" means for the shape of a function. In particular, if you know a function increases towards an asymptote, you know it must become concave down, while a function decreasing towards an asymptote must become concave up.

3 The basic tasks

When we ask you to "sketch" a function in this class, we generally mean that you should perform the following tasks. Each task gives some clue about what the graph must eventually look.

- Identify the domain. To do this, look at what would possibly make the definition of the function fail to work – are there inputs where you would need to divide by zero? Where you would attempt to take the logarithm of a negative number?

- Classify the discontinuities. This amounts to computing limits – if you find that the function is discontinuous at a point, find the limits on the left and right.

    - If both one-sided limits exist and are equal, this discontinuity is removable.
    - If both one-sides limits exist but are different, this is a jump discontinuity.
    - If one or both one-sided limits is infinite, this is a vertical asymptote.
    - If none of these cases holds, it is a more exotic discontinuity (we are unlikely to see many examples of these).

- Find all local extrema, and identify where the function is increasing and where it is decreasing. Usually, you perform this by computing the derivative, and checking where it is either zero or undefined (the critical numbers), and then checking where it is positive and where it is negative (if possible).

- (Often omitted) Find where the function is concave up and where it is concave down. To do this, find the second derivative and see where it is positive and negative; the points where it changes concavity are called inflection points. This step is often omitted since it may be computationally difficult to get the second derivative (you don't need to do it unless we ask for it).

- Take the limits at infinity to find horizontal asymptotes.

Examples of these tasks are shown in the examples that follow.

There is also one additional task, which we'll rarely ask for (if ever), but is worth mentioning: sometimes you can find "slant asymptotes" as well; we mention these in the last section but won't spend much time on them.

4   Examples

Example 4.1. Sketch the graph of $f(x) = e^{-x^2}$, including finding the concavity and inflection points.

Solution. The **domain** of this function consists of all real numbers; there are no discontinuities. The first derivative is $f'(x) = e^{-x^2}\frac{d}{dx}(-x^2) = -2xe^{-x^2}$ (by the chain rule). Since e^{-x^2} is positive for any number x (e to any power is positive), $-2xe^{-x^2}$ always has the same sign as $-2x$: it is positive when $x < 0$, negative when $x > 0$, and zero precisely when $x = 0$. So there is one critical number: $x = 0$. At this number, $f(x)$ switches from increasing to decreasing, so this is a local maximum. The value there is $f(0) = 1$.

Next, look at the second derivative. By the product rule and the chain rule,

$$
\begin{aligned}
f''(x) &= \frac{d}{dx}(-2x)e^{-x^2} - 2x\frac{d}{dx}e^{-x^2} \\
&= -2e^{-x^2} - 2x(-2x)e^{-x^2} \\
&= -2e^{-x^2} + 4x^2e^{-x^2} \\
&= 2e^{-x^2}\left(2x^2 - 1\right).
\end{aligned}
$$

Again using the fact that $e^{-x^2} > 0$ for all x, we see that the sign of $f''(x)$ is the same as the sign of $2x^2 - 1$. So it is positive when $|x| > \frac{1}{\sqrt{2}}$ and negative when $|x| < \frac{1}{\sqrt{2}}$; it is zero when $x = \pm\frac{1}{\sqrt{2}} \approx \pm 0.7$. So there are inflection points at $x = \pm\frac{1}{\sqrt{2}}$; at both of these the value of the function is $f(\pm\frac{1}{\sqrt{2}}) = e^{-1/2} \approx 0.6$. The function is concave up until the first inflection point, then concave down until the second, then concave up again.
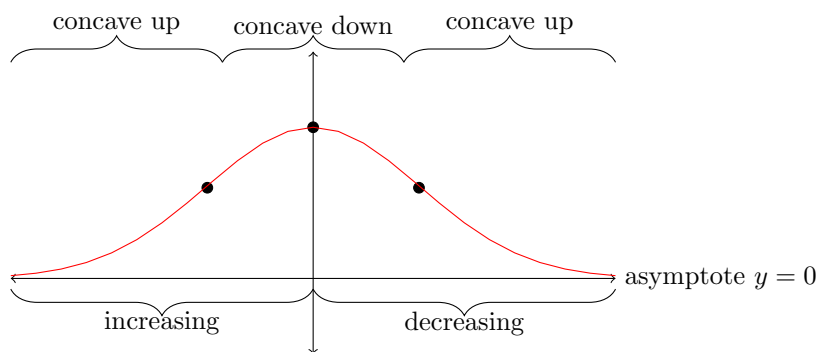
Finally, we can find the horizontal asymptotes:

$$\lim_{x \to \pm\infty} e^{-x^2} \quad = \quad e^{-(\pm\infty)^2}$$
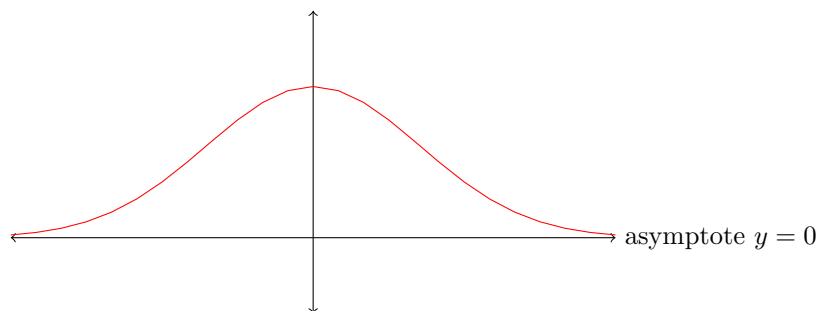$$= \quad e^{-\infty}$$
$$= \quad 0$$

So both ends have horizontal asymptote $y = 0$. All of this information is summarized below.
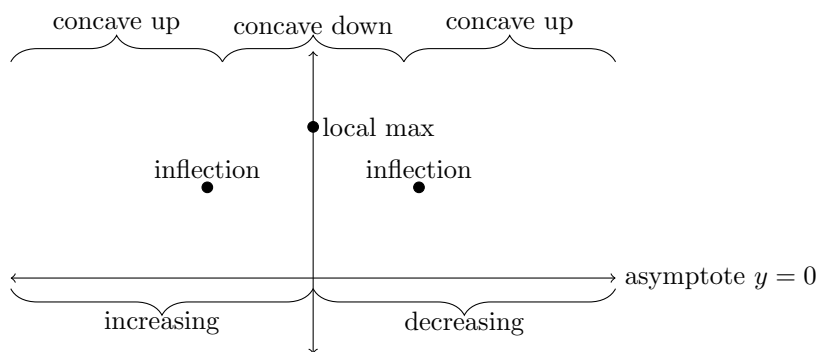
This is enough information to make a pretty decent sketch. The graph is shown below.

Or, without all the stray marks:

This curve is the famous *bell-shaped curve*, which is ubiquitous in statistics. It emerged in nature whenever some outcome depends on the average of many independent events.

Example 4.2. Sketch $f(x) = \dfrac{x^2}{x+1}$.

Solution. I'll start be writing down the first derivative.

$$
\begin{aligned}
f(x) &= \frac{x^2}{x+1} \\
f'(x) &= \frac{2x \cdot (x+1) - x^2 \cdot 1}{(x+1)^2} \\
&= \frac{x^2 + 2x}{(x+1)^2} \\
&= \frac{x(x+2)}{(x+1)^2}
\end{aligned}
$$

The function has only one discontinuity: at $x = -1$ (where you would need to divide by 0. You can compute the limits on either side as follows.

$$
\begin{aligned}
\lim_{x \to (-1)^-} \frac{x^2}{x+1} &= \frac{1}{0^-} \\
&= -\infty \\
\lim_{x \to (-1)^+} \frac{x^2}{x+1} &= \frac{1}{0^+} \\
&= \infty
\end{aligned}
$$

The function has critical numbers where the derivative is zero: $x = 0$ and $x = -2$. You can see that the sign of $f'(x)$ is the same as the sign of $x(x+2)$ (since the denominator is always positive when it isn't zero,being a square of a number). The sign of $x(x+2)$ is positive, then negative between -2 and 0, then positive again. So there is a local max at $(-2, f(-2)) = (-2, -4)$ and a local minimum at $(0, f(0)) = (0, 0)$. So so far we know the following about this graph.

- The function increases to a local maximum at $(-2, -4)$.

- Then the function decreases to a vertical asymptote, going to $-\infty$ near $x = -1$.

- From $x = -1$ to $x = 0$, the function comes back from ∞ and decreases to $(0, 0)$.

- After $x = 0$, the function increases.

This information is enough to draw a pretty good sketch, though not a perfect one. A much better sketch is possible if the function is rewritten in a certain way; we'll see this in the next section. Here is a precise plot.
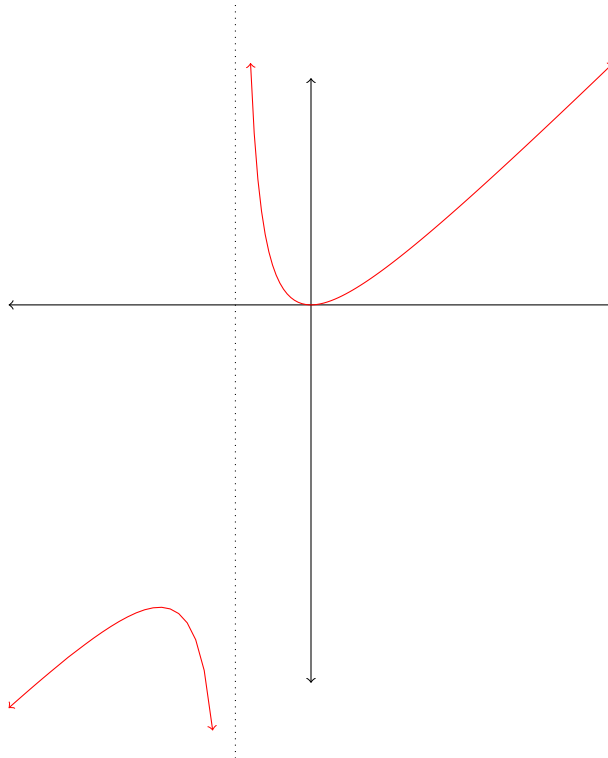
This graph actually has an unusual feature called a slant asymptote (in this case, the asymptote is $y = x - 1$); we'll look at this notion (in this particular example) in the last section.

Example 4.3. Sketch $f(x) = \dfrac{e^x + 1}{e^x - 1}$.

Solution. There is discontinuity where the denominator is 0, i.e. where $e^x = 1$. There is only one such x: $x = 0$. The limits near this discontinuity are:

$$\begin{aligned}
\lim_{x \to 0^-} \frac{e^x + 1}{e^x - 1} &= \frac{2}{0^-} \\
&= -\infty \\
\lim_{x \to 0^+} \frac{e^x + 1}{e^x - 1} &= \frac{2}{0^+} \\
&= \infty
\end{aligned}$$

The limits at infinity are:

$$\lim_{x \to -\infty} \frac{e^x + 1}{e^x - 1} = \frac{0 + 1}{0 - 1}$$
$$= -1$$
$$\lim_{x \to \infty} \frac{e^x + 1}{e^x - 1} = \lim_{x \to \infty} \frac{e^x + 1}{e^x - 1} \cdot \frac{1/e^x}{1/e^x}$$
$$= \lim_{x \to \infty} \frac{1 + 1/e^x}{1 - 1/e^x}$$
$$= \frac{1 + 0}{1 + 0}$$
$$= 1$$

The derivative of the function is:

$$
\begin{aligned}
f'(x) &= \frac{e^x(e^x - 1) - (e^x + 1)e^x}{(e^x - 1)^2} \\
&= \frac{e^{2x} - e^x - e^{2x} - e^x}{(e^x - 1)^2} \\
&= -\frac{2e^x}{(e^x - 1)^2}
\end{aligned}
$$

Since squares are always positive, the denominator of $f'(x)$ is always positive. The numerator $-2e^x$ is always negative. So $f'(x)$ is always negative where it is defined, so the function is always decreasing (except at its asymptote).

So from this, we know the following information about the graph.

- It comes in from a horizontal asymptote at $y = -1$, then decreases toward $-\infty$ as x approaches 0 from the left.

- It returns from $+\infty$ on the right of $x = 0$, then decreases toward a horizontal asymptote at $y = 1$.

This is enough information to make a pretty good sketch.
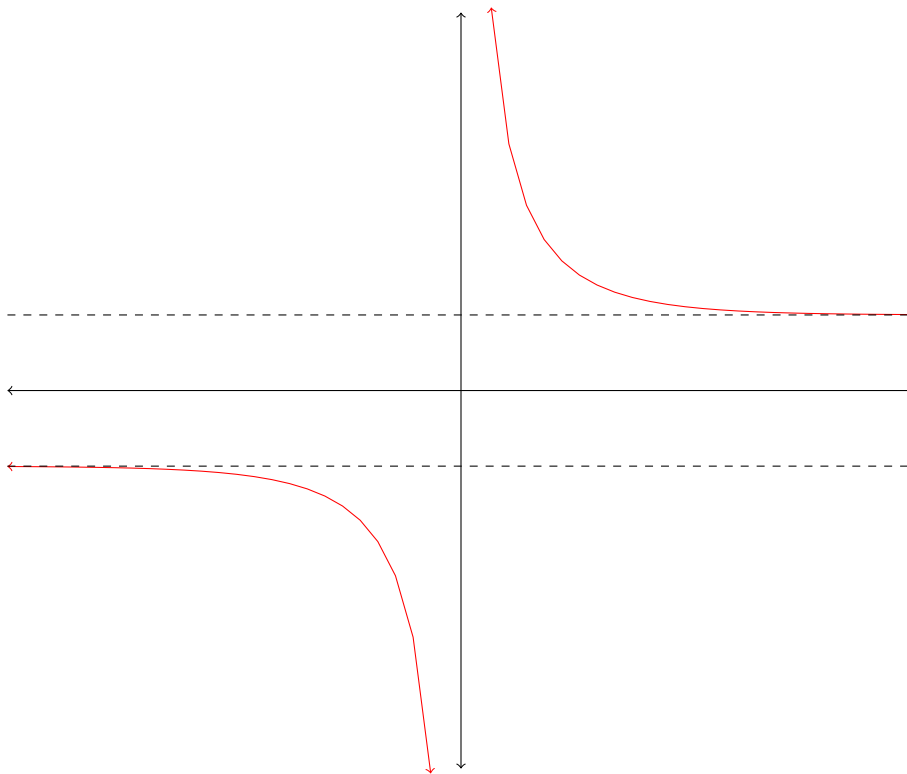
Example 4.4. Sketch $f(x) = \left(\dfrac{2e^x - 1}{e^x + 1} \right)^2$.

Solution. The denominator is always positive, so there are no discontinuities.
The derivative can be found with the chain and quotient rules.

$$
\begin{aligned}
f'(x) &= 2\left(\frac{2e^x - 1}{e^x + 1} \right) \frac{d}{dx}\left(\frac{2e^x - 1}{e^x + 1} \right) \\
&= 2\left(\frac{2e^x - 1}{e^x + 1} \right) \frac{2e^x(e^x + 1) - (2e^x - 1)e^x}{(e^x + 1)^2} \\
&= 2\left(\frac{2e^x - 1}{e^x + 1} \right) \frac{3e^x}{(e^x + 1)^2}
\end{aligned}
$$

It looks difficult to do anything with this, but it actually is not so bad: everything in sight here is always positive, except the factor $(2e^x - 1)$. This is zero only when $2e^x = 1$, i.e. $x = \ln \frac{1}{2} = -\ln 2 \approx 0.7$. For smaller x, $2e^x < 1$, so $f'(x) < 0$; for larger x $f'(x) > 0$. So $f(x)$ is decreasing until a local minimum at $(-\ln 2, f(-\ln 2)) = (-\ln 2, 0)$, and increasing afterwards.

We'll skip computing the concavity, since it would be pretty painful in this case.

To see the behavior in the limit, we can compute the limits at infinity as follows.

$$\begin{aligned}
\lim_{x \to -\infty} \left(\frac{2e^x - 1}{e^x + 1} \right)^2 &= \left(\frac{0-1}{0+1} \right)^2 \\
&= (-1)^2 \\
&= 1 \\
\lim_{x \to \infty} \left(\frac{2e^x - 1}{e^x + 1} \right)^2 &= \lim_{x \to \infty} \left(\frac{2e^x - 1}{e^x + 1} \cdot \frac{1/e^x}{1/e^x} \right)^2 \\
&= \lim_{x \to \infty} \left(\frac{2 - 1/e^x}{1 + 1/e^x} \right)^2 \\
&= \left(\frac{2-0}{1+0} \right)^2 \\
&= 4
\end{aligned}$$

So this function comes from a horizontal asymptote at $y = 1$, decreases to a local minimum around $(-0.6, 0)$, the increases to a horizontal asymptote at $y = 4$. This is enough to draw a decent sketch. Here is the graph.
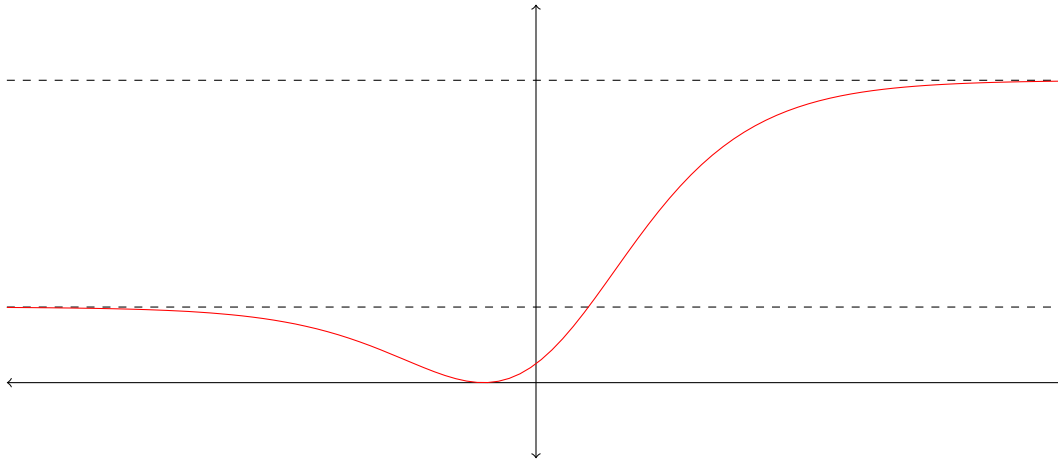
5   Slant asymptotes

There is another interesting qualitative feature that we won't discuss at length, but I'll show in one example. Consider the function $f(x) = \frac{x^2}{x+1}$, which we sketched above. This function can be rewritten by performing "long division": one can compute (using an algorithm essentially identical to long division of numbers) that x^2 divided by $(x+1)$ gives $(x-1)$ with a remainder of 1. More precisely,

$$\frac{x^2}{x+1} = (x-1) + \frac{1}{x+1}$$

This form actually tells much more about the long-term behavior about $f(x)$, because as x grows to either ∞ or $-\infty$, the term $\frac{1}{x+1}$ goes to 0. So the function becomes essentially the same as $x - 1$. In fact, you can see this in the plot.