Lecture 19: Limits to infinity

Nathan Pflueger

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1 Introduction

This is the second part of our discussion of limits involving infinity. In this lecture, we consider limits as x goes to infinity (or minus infinity). The techniques are very similar to what we saw last time: we compute this limits by attempting to "plug in infinity" into the function. In many cases, it will be necessary to modify the function in some way or another to get something that is well-defined. Later, we'll discuss a more systematic technique, l'Hôpitals rule, for computing limits like this, but it is worth seeing more direct methods first (the more direct methods tend to be easier to apply, but sometimes require a bit more cleverness to find).

The reference for today is Stewart §2.5 (same as last time).

2 What is a limit at infinity?

I'll start by writing down a couple obvious examples of limits at infinity. These will illustrate what, exactly, a limit at infinity means.

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{x} &= 0\\ \lim_{x\to-\infty}\frac{1}{x} &= 0\end{aligned}$$

Here's a qualitative way to describe what these two limits mean: if x is chosen *very large*, then $\frac{1}{x}$ will be very small. In fact, if x is *large enough*, the number $\frac{1}{x}$ will fall within *any possible margin of error* of 0.

This suggest the following quasi-formal definition of a limit at infinity.

$\lim_{x\to\infty} f(x) = L$ means that any measuring instrument, no matter how accurate, will record $f(x) = L$ for once x gets large enough.

Example 2.1. Suppose that the amount of a certain radioactive material in a substance after t years is e^{-t} grams. If we have a single measuring instrument, which is only accurate to within 0.001 grams, then sooner or later, this measuring instrument will claim that there are 0 grams of the radioactive material left. There may still be trace amounts, that could be detected by a more accurate instrument. But that instrument, too, will eventually claim that there are 0 grams left (since it has some margin of error as well). This is what it means the $\lim_{t\to\infty} e^{-t} = 0$.

In applications, the "limit at infinity" basically means the "behavior after a long time." In mathematics, however, we can't just say "a long time," since we're not allowed to have any opinions about how accurate is accurate enough. So instead we use the symbol ∞. If you like, this symbol means "a number that is always large enough."

This same sort of thinking is also a good way to explain what it means for a limit to equal infinity. For example, consider the following fact.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

This means that $\frac{1}{x}$ will be "large enough" for any particular purpose, as long as you can get x to be close enough to 0. Here's a physical example.

Example 2.2. The ideal gas law states (among other things) that if a certain quantity of an ideal gas[1] is held at constant temperature, then the product of pressure and volume is constant; that is $PV = C$, where P is the pressure, V is the volume, and C is some constant. This means that $P = \frac{C}{V}$. Now think about what happens if you try to shrink the volume to 0: $\lim_{V \to 0^+} \frac{C}{V} = \infty$. The physical meaning of this is: if you compress the gas small enough (but keep temperature constant), then the pressure will become arbitrarily large. No matter how large you want the pressure to be, it will be that large once the gas is compressed to sufficiently small volume.

Now that we have some informal ways to think about what ∞ means, we'll see some techniques to compute limits at infinity.

3 Examples

The following examples illustrate the main computational tools that are helpful in evaluating limits at infinity. The methods we'll see today follow the same general form.

- First try naively substituting ∞ or $-\infty$ into the expression for x (more formally, this is shorthand for taking limits of each part individually, e.g. taking the limit of the numerator and denominator separately).
- If this first step gives an unambiguous result, we're done. Otherwise, it may give something like $0/0$ or ∞/∞. In this case, we *modify the original expression* (most commonly by multiplying by some factor in the numerator and denominator) and try again.

Later we'll learn another frequently-useful trick called l'Hôpital's rule; for now we'll stick to elementary methods.

We will compute the following limits as examples.

1. $\displaystyle\lim_{x \to -\infty} \frac{x^2}{x+1}$
2. $\displaystyle\lim_{x \to \infty} \frac{x^2}{x+1}$
3. $\displaystyle\lim_{x \to -\infty} \frac{2e^x - 1}{e^x + 1}$
4. $\displaystyle\lim_{x \to \infty} \frac{2e^x - 1}{e^x + 1}$
5. $\displaystyle\lim_{x \to \infty} \frac{x^2 - 3x + 2}{x^2 + 3x + 2}$
6. $\displaystyle\lim_{x \to \infty} \frac{\cos x}{x}$

[1] Of course no gas is actually "ideal," but we're speaking in broad strokes here.

7. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x-7}$

8. $\displaystyle\lim_{x\to-\infty} \left(\sqrt{x^2+1}-x\right)$

9. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+1}-x\right)$

Solutions.

1. Naive substitution would give $\displaystyle\lim_{x\to-\infty} \frac{x^2}{x+1} = \frac{(-\infty)^2}{\infty} = \frac{\infty}{-\infty}$, which is not determinate. So we must perform some modification. In this case, we can transform the denominator to give a finite limit, by scaling it with a factor of x, as follows.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2}{x+1}\cdot\frac{1/x}{1/x} &= \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x}} \\
&= \frac{-\infty}{1+\frac{1}{-\infty}} \\
&= \frac{-\infty}{1+0} \\
&= -\infty
\end{aligned}$$

2. The same modification will work for the limit in this direction.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2}{x+1}\cdot\frac{1/x}{1/x} &= \lim_{x\to\infty} \frac{x}{1+\frac{1}{x}} \\
&= \frac{\infty}{1+\frac{1}{\infty}} \\
&= \frac{\infty}{1+0} \\
&= \infty
\end{aligned}$$

3. Naive substitution works here: $\displaystyle\lim_{x\to-\infty} \frac{2e^x-1}{e^x+1} = \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{-1}{1} = -1$. Remember as usual that notation like $e^{-\infty}$ is really just shorthand for $\displaystyle\lim_{x\to-\infty} e^x$.

4. Naive substitution does *not* work here: it would yield $\frac{\infty}{\infty}$. In this case, we can tame both the numerator and denominator by multiplying them by $1/e^x$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2e^x-1}{e^x+1} &= \lim_{x\to\infty} \frac{2e^x-1}{e^x+1}\cdot\frac{1/e^x}{1/e^x} \\
&= \lim_{x\to\infty} \frac{2-1/e^x}{1+1/e^x} \\
&= \frac{2-\frac{1}{\infty}}{1+\frac{1}{\infty}} \\
&= \frac{2-0}{1-0} \\
&= 2
\end{aligned}$$

5. Here, prevent the indeterminate form ∞/∞ by dividing on the top and bottom by the function x^2. This function is chosen because it is "roughly" the rate of growth of both the numerator and denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2-3x+2}{x^2+3x+2} \cdot \frac{1/x^2}{1/x^2} &= \lim_{x\to\infty} \frac{1-3/x+2/x^2}{1+3/x+2/x^2} \\
&= \frac{1-0+0}{1+0-0} \\
&= 1
\end{aligned}$$

Note that the limit as $x \to -\infty$ is the same: it is also 1. So the horizontal line $y = 1$ is an asymptote in both directions.

6. As $x \to \infty$, the denominator grows arbitrarily large, while the numerator oscillates between -1 and 1. We cannot write a limit for $\cos x$ as $x \to \infty$, but we don't need to: since it is bounded between -1 and 1 for all time, and we are dividing by arbitrarily large numbers, the quotient becomes arbitrarily small as x grows. So the limit must be 0. Here is the notation I like to use to succinctly express this argument.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\cos x}{x} &= \frac{\text{(bounded)}}{\infty} \\
&= 0
\end{aligned}$$

The more formal way to establish this limit is to use the squeeze theorem: $\frac{-1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$; the limits of both bounding functions are 0, so therefore this must also be the limit of the bounded function.

7. The numerator is no a polynomial, but you should expect it to grow about as fast as $\sqrt{x^2} = x$ (warning: $\sqrt{x^2}$ is only equal to x when $x > 0$; otherwise it is equal to $-x$, since the square root is always a positive number). So modify the function by dividing by x on the top and bottom, as follows.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x-7} \frac{1/x}{1/x} &= \frac{\sqrt{(x^2+1)/x^2}}{2-7/x} \\
&= \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2-7/x} \\
&= \frac{\sqrt{1+0}}{2-0} \\
&= \frac{1}{2}
\end{aligned}$$

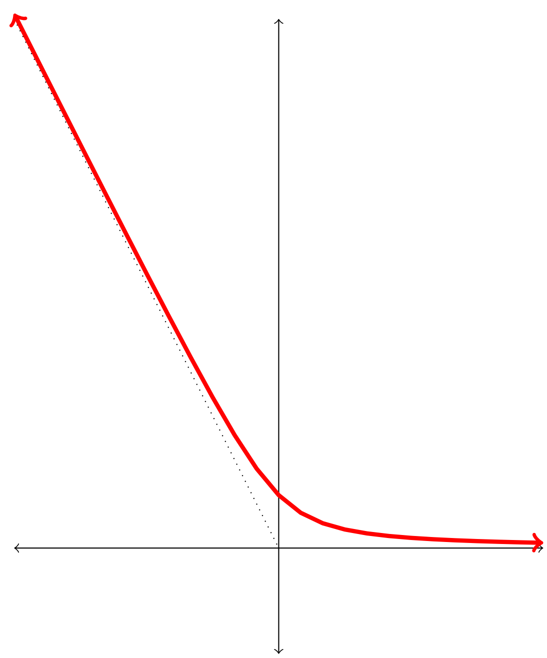
Note. If you evaluate the limit instead as $x \to -\infty$ (as we did in class in my section on Thursday), there is a small wrinkle: when x is negative, the correct substitution is $\sqrt{x^2+1} \cdot 1/x = -\sqrt{1+1/x^2}$. The reason for this is that $1/x$ is *negative* in this case, so $1/x = -\sqrt{1/x^2}$. So the resulting limit is $-\frac{1}{2}$. This fact (that a x is only $\sqrt{x^2}$ if $x > 0$) is important to remember in many situations, but it isn't the concept we're trying to emphasize at the moment, so I decided to write the simpler version ($x \to \infty$) in the notes.

8. Evaluate naively: $\lim_{x\to-\infty} \left(\sqrt{x^2+1} - x\right) = \infty - (-\infty) = \infty + \infty = \infty$.

9. Naive evaluation would give $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\infty-\infty$, which is not defined. So some modification is needed. We've seen in other contexts that expressions involving square roots are often usefully transformed by multiplying by a "conjugate" expression (we called this procedure "rationalizing the numerator" before). This modification allows us to evaluate the limit in this case as follows.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} &= \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} \\
&= \frac{1}{\sqrt{x^2+1}+x} \\
&= \frac{1}{\infty+\infty} \\
&= \frac{1}{\infty} \\
&= 0
\end{aligned}$$

The function we have just studied has the following somewhat interesting graph.



It has what is called a *slant asymptote* of $y=-2x$ for negative values of x, while it has a horizontal asymptote of $y=0$ for $x>0$. If you are skilled with algebra, see if you can show clearly why the function begins to hug closely to the line $y=-2x$ for large negative values of x.

4 Summary of techniques

In most cases of the examples above, we resolved an indeterminate form by **canceling the fastest-growing part** of the numerator or denominator of a function. Generally speaker, it seems to work best to first identify whether the numerator or denominator is growing faster, and then cancel the fastest-growing part. This technique usually works best to deal with a form that is giving ∞/∞.

In some cases, this method will not work; in the last limit we attempted to compute, we had the indeterminate form $\infty - \infty$. In this case, we needed to perform a different modification, namely "rationalizing the numerator." This modification is an example of the fact that there are many other ways to modify the expression of a function; different ideas are useful in different situations.