Lecture 18: Infinite limits

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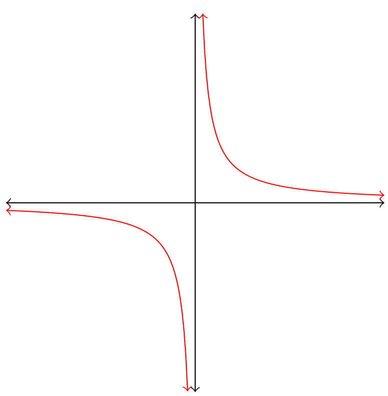
1 Introduction

In this lecture and the next, we return to limits briefly to expand our vocabulary a little bit. Specifically, we will consider asymptotes, which are places where the graph begins to hug closer and closer to some line. This lecture considers vertical asymptotes: those where the graph hugs near a vertical line. The next lecture will concern horizontal asymptotes. The vocabulary we will introduce here involves the notion of infinity, denoted ∞. Strictly speaking, ∞ is not a number, but it is a concept that is useful for describing certain limit processes. We will discuss what it means for a limit to be ∞ or $-\infty$, and see how to determine this fact in various examples. Note that when a limit "equals ∞" we will still say "it doesn't exist," since we won't think of ∞ as an honest number.

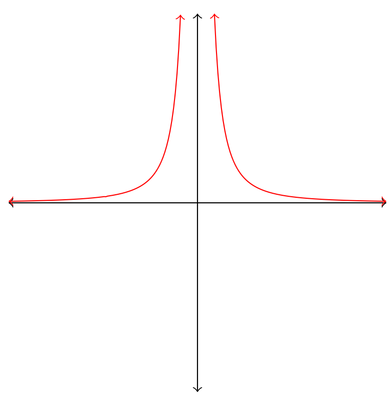
The reference for this lecture (and the next) is Stewart §2.5.

2 Vertical asymptotes

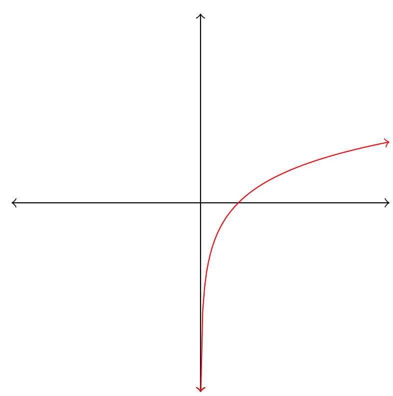
We've had several occasions to encounter a certain specific type of singularity, called a *vertical asymptote*. The following three functions are common examples of this type of behavior; all have a vertical asymptote at $x = 0$.



$y = 1/x$



$y = 1/x^2$



$y = \ln x$

These three examples all show the same basic fact: the function is discontinuous at $x = 0$ because it blows up nearby. But they do have different behavior, since they blow up in different directions.

We now introduce the following notation to distinguish situations like this.

- $\lim_{x \to c^+} f(x) = \infty$ will mean that $f(x)$ becomes arbitrarily large for values $x > c$ sufficiently close to c.
- $\lim_{x \to c^+} f(x) = -\infty$ will mean that $f(x)$ becomes arbitrarily negative for values $x > c$ sufficiently close to c.

We will use similar definitions for $\lim_{x\to c^-} f(x)$ being ∞ or $-\infty$.

If we write $\lim_{x\to c} = \infty$ (or $-\infty$), this will mean that *both* one sided limits are ∞ (or $-\infty$). In these examples:

- $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.
- $\lim_{x\to 0^+} \frac{1}{x} = \infty$.
- $\lim_{x\to 0} \frac{1}{x^2} = \infty$ (from both sides).
- $\lim_{x\to 0^+} \ln x = -\infty$.
- $\lim_{x\to 0^-} \ln x$ has no meaning (since $\ln x$ is undefined for $x < 0$).

Warning. Even when we conclude that a limit is ∞ or $-\infty$, we will *still say* "it does not exist." So we are not treating ∞ and $-\infty$ like real numbers in any sense; they're really just shorthand for a particular kind of discontinuity.

3 Arithmetic of infinite limits

The following are some **informal** rules for computing limits involving infinity. After I state them, I'll give a more formal version of each statement.

- $\frac{1}{0^+} = \infty$
- $\frac{1}{0^-} = -\infty$
- $\frac{1}{\infty} = 0$
- $\infty + \infty = \infty$
- $\infty \cdot \infty = \infty$

There are also several variants of these, such as $-\infty - \infty = -\infty$, and $(-\infty)\cdot(-\infty) = \infty$.

When I write the symbol 0^+, what I really mean is "the limit of positive numbers going to 0." To see what I mean by this, here is a basic example. There will be more complex examples in the next section.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1}{x} &= \frac{1}{0^+} \\
&= \infty \\
\lim_{x\to 0^-} \frac{1}{x} &= \frac{1}{0^-} \\
&= -\infty
\end{aligned}$$

If you like you can think of 0^+ as meaning "an infinitesimally small, but positive, number," while 0^- stands for "an infinitesimally small, but negative, number."

Warning: Students are often tempted to do much more arithmetic with ∞ than is possible. For example, all three of the following expressions **cannot have any meaning:**

$$\frac{0}{0}$$
$$\frac{\infty}{\infty}$$
$$\infty - \infty$$

These are called *indeterminate forms.* To get a sense for why they can't have any good meaning, consider $0/0$. It is tempting to say it should be 1. But notice that $0/0$ should also equal $(5 \cdot 0)/0$, which suggests that $0/0$ should also equal 5. Similar remarks can be made for the other two expressions. We'll have more to say about indeterminate forms later; they are often treated using a technique called l'Hôpital's rule.

I'll finish this section by giving precise and correct versions of the informal rules above. In each of these cases, you can replace the limits in the "formal version" by 1-sided limits, and the statements will still be true. The only informal part of the sentences on the right is the word "x around c," which means "any x within *some* margin of error of c, but not exactly equal to c" For one-sided limits, "around" would only refer to values of x on one side or the other.

Informal rule	Formal version
$\frac{1}{0^+} = \infty$	If $\lim_{x\to c} f(x) = 0$ and $f(x) > 0$ for x around c, then $\lim_{x\to c} \frac{1}{f(x)} = \infty$.
$\frac{1}{0^-} = -\infty$	If $\lim_{x\to c} f(x) = 0$ and $f(x) < 0$ for x around c, then $\lim_{x\to c} \frac{1}{f(x)} = \infty$.
$\frac{1}{\infty} = 0$	If $\lim_{x\to c} f(x) = \infty$, then $\lim_{x\to c} \frac{1}{f(x)} = 0$.
$\infty + \infty = \infty$	If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then $\lim_{x\to c} (f(x) + g(x)) = \infty$.
$\infty \cdot \infty = \infty$	If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then $\lim_{x\to c} (f(x) \cdot g(x)) = \infty$.

4 Computing infinite limits

This section will illustrate how to compute infinite limits with a sequence of examples.

For each of the following functions, identify all discontinuities. For each vertical asymptote, identify each one-sided limit as either ∞, $-\infty$, or neither.

1. $\dfrac{x^2 - 3x + 2}{x + 1}$
2. $\dfrac{e^x}{x^2 + 4x + 4}$
3. $\dfrac{x^2 - 13x + 42}{x^2 - 12x + 36}$
4. $\dfrac{e^x}{1 - \cos x}$
5. $\dfrac{x^2 - 8x + 7}{x^2 - 4x + 5}$
6. $e^{1/x}$
7. $\dfrac{x^3 - 7x + 7}{x^2 - 3x + 2}$
8. $\dfrac{\cos x}{\ln(x^2 + 1)}$

Solutions.

1. This is a quotient of two continuous functions, so the discontinuities occur precisely when the denominator is 0. So the only discontinuity is at $x = -1$. When $x = -1$, the numerator of this function is nonzero. So there is a vertical asymptote. Now compute the actual limits (on both sides) more precisely. To do this, look at the *sign* of the denominator: the denominator $(x+1)$ is negative for $x < -1$, then positive for all $x > -1$. Therefore:

$$\begin{aligned}
\lim_{x\to(-1)^-} \frac{x^2-3x+2}{x+1} &= \frac{6}{0^-} \\
&= -\infty \\
\lim_{x\to(-1)^+} \frac{x^2-3x+2}{x+1} &= \frac{6}{0^+} \\
&= \infty
\end{aligned}$$

2. The only discontinuities of $\frac{e^x}{x^2+4x+4}$ will occur where the denominator is equal to 0. This occurs when $x^2+4x+4=0$, i.e. $(x+2)^2=0$, so the only discontinuity is at $x=-2$. The denominator is a square, so it is never negative. The numerator e^x is always nonzero and positive. Therefore the two one-sided limits at the singular number $x=-2$ are:

$$\begin{aligned}
\lim_{x\to(-2)^-} \frac{e^x}{(x+2)^2} &= \frac{e^{-2}}{0^+} \\
&= \infty \\
\lim_{x\to(-2)^+} \frac{e^x}{(x+2)^2} &= \frac{e^{-2}}{0^+} \\
&= \infty
\end{aligned}$$

   In both these cases, I have written "0^+" to indicate that the limit of $(x+2)^2$ is zero, and all nearby values are positive. This allows me to conclude that the limit of the quotients is positive infinity, rather than negative infinity. Since both one-sided limits are equal, we can also say that the two-sided limit is ∞.

$$\lim_{x\to(-2)} \frac{e^x}{(x+2)^2} = \infty$$

3. The only discontinuities are where the denominator is 0 that is, where

$$\begin{aligned}
x^2-12x+36 &= 0 \\
(x-6)^2 &= 0 \\
x &= 6
\end{aligned}$$

   So there is exactly one discontinuity: at $x=6$. At this value, the numerator is $(6)^2-13(6)+42=0$. Therefore if we try to take limits of the top and bottom separately, we get the indeterminate form $0/0$. More work is needed to tell what kind of singularity this is.

   In this case, there is a good way to rewrite the function: by factoring it.

$$
\begin{aligned}
\frac{x^2 - 13x + 42}{x^2 - 12x + 36} &= \frac{\cancel{(x-6)}(x-7)}{(x-6)^{\cancel{2}}} \\
&= \frac{x-7}{x-6}
\end{aligned}
$$

Once the function is re-expressed this way, we can compute the two one-sided limits and see that this discontinuity is, in fact, a vertical asymptote. The numerator approaches $6 - 7 = -1$ at $x = 6$, while the denominator is negative for $x < 6$ and positive for $x > 6$. Therefore:

$$
\begin{aligned}
\lim_{x \to 6^-} \frac{x^2 - 13x + 42}{x^2 - 12x + 36} &= \lim_{x \to 6^-} \frac{x-7}{x-6} \\
&= \frac{-1}{0^-} \\
&= +\infty \\
\lim_{x \to 6^+} \frac{x^2 - 13x + 42}{x^2 - 12x + 36} &= \lim_{x \to 6^+} \frac{x-7}{x-6} \\
&= \frac{-1}{0^+} \\
&= -\infty
\end{aligned}
$$

So the function goes up to ∞ from the left, and then comes back from $-\infty$ on the right.

4. The function $\dfrac{e^x}{1 - \cos x}$ is again a quotient of continuous functions, so its only discontinuity is at $x = 0$, where the denominator is 0. The denominator is 0 precisely when $\cos x = 1$, and this occurs precisely when x is an integer multiple of 2π, i.e. a number $2\pi n$, where n **is an integer.** Notice that $1 - \cos x$ is never negative, since $\cos x \leq 1$, so informally speaking, $\lim_{x \to 2\pi n} \cos x = 0^+$, and we can compute at each discontinuity:

$$
\begin{aligned}
\lim_{x \to 2\pi n} \frac{e^x}{1 - \cos x} &= \frac{e^{2\pi n}}{0^+} \\
&= \infty
\end{aligned}
$$

Here we've also used the fact that $e^{2\pi n}$ is always positive.

So this function has a discontinuity at each number $0, \pm 2\pi, \pm 4\pi, \cdot$, and its limit is ∞ (from both sides) at each of these discontinuities.

5. The function $\dfrac{x^2 - 8x + 7}{x^2 - 4x + 5}$ is a quotient of two continuous functions, so it will be discontinuous only where the denominator is 0. If you try to solve $x^2 - 4x + 5$ with the quadratic formula, you will obtain $x = 2 \pm \sqrt{-1}$, so there are no real solutions. So in fact this function is **continuous everywhere**.

6. The function $e^{1/x}$ is continuous everywhere that $\frac{1}{x}$ is continuous (since e^x is continuous, and a composition of continuous functions is continuous). Therefore the only discontinuity is at $x = 0$. We can compute the two one-sides limits (again, using slightly informal notation) as follows.

$$
\begin{aligned}
\lim_{x\to 0^-} e^{1/x} &= e^{\lim_{x\to 0^-} \frac{1}{x}} \\
&= e^{-\infty} \\
&= 0 \\
\lim_{x\to 0^+} e^{1/x} &= e^{\lim_{x\to 0^+} \frac{1}{x}} \\
&= e^{\infty} \\
&= \infty
\end{aligned}
$$

So this function exhibits a rather curious type of behavior: the limit from one side exists, while the limit from the other side blows up to ∞.

Aside: remember that the notation I've used above is not quite precise, but it can be made precise. When I write $e^{-\infty}$, I don't mean that you can literally plug the "number" $-\infty$ into the function e^x. Really, this is just shorthand for "a limit of value of e^x, where the input gets arbitrarily small." The same is true elsewhere when I write things like 0^+ and 0^-: I don't mean these to literally be numbers, but rather as shorthand for a limiting process.

7. The function $\dfrac{x^3 - 7x + 7}{x^2 - 3x + 2}$ is a quotient of two continuous functions, so it has discontinuities when the denominator is 0. The denominator is 0 when $x^2 - 3x + 2 = 0$. This equation can be solved by factoring: it is equivalent to $(x-1)(x-2) = 0$, so the denominator is 0 precisely when $x = 1$ or $x = 2$. The numerator turns out to be nonzero at both these values. So we can find all of the one-sided limits as follows.

$$
\begin{aligned}
\lim_{x\to 1^-} \frac{x^3 - 7x + 7}{x^2 - 3x + 2} &= \frac{1 - 7\cdot 1 + 7}{0^-} \\
&= \frac{1}{0^+} \\
&= \infty \\
\lim_{x\to 1^+} \frac{x^3 - 7x + 7}{x^2 - 3x + 2} &= \frac{1 - 7\cdot 1 + 7}{0^+} \\
&= \frac{1}{0^-} \\
&= -\infty
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to 2^-} \frac{x^3 - 7x + 7}{x^2 - 3x + 2} &= \frac{2^3 - 7\cdot 2 + 7}{0^+} \\
&= \frac{1}{0^-} \\
&= -\infty \\
\lim_{x\to 2^+} \frac{x^3 - 7x + 7}{x^2 - 3x + 2} &= \frac{2^3 - 7\cdot 2 + 7}{0^-} \\
&= \frac{1}{0^+} \\
&= \infty
\end{aligned}
$$

So there are two vertical asymptotes: one at $x = 1$ and one at $x = 2$. At the first, the function blows up to ∞ and comes back from $-\infty$, while at the second the function dives down to $-\infty$ and returns from ∞.

8. The numerator of $\dfrac{\cos x}{\ln(x^2+1)}$ is discontinuous only when the denominator is undefined or 0. It is always defined, since $x^2 + 1$ is always positive, so $\ln(x^2 + 1)$ is always defined. It is equal to 0 when:

$$\begin{aligned}
\ln(x^2+1) &= 0 \\
x^2+1 &= e^0 \\
x^2+1 &= 1 \\
x^2 &= 0 \\
x &= 0
\end{aligned}$$

So there is a single discontinuity at $x = 0$. Around this discontinuity, the denominator is, in fact, always positive: this is because $\ln(x^2 + 1) > 0$ whenever $x^2 + 1 > 1$, i.e. $x^2 > 0$, but this is true whenever $x \neq 0$. Therefore we can actually see that the two-sided limit is ∞ at this discontinuity, as follows.

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x}{\ln(x^2+1)} &= \frac{\cos(0)}{0^+} \\
&= \frac{1}{0^+} \\
&= \infty
\end{aligned}$$