Lecture 14: Products, quotients, and trig functions

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1 Introduction

This lecture is essentially a catalog of several more rules that can be used to efficiently differentiate a larger family of functions (without appeal to the limit definition). We will discuss how to differentiate a product or quotient of any two functions (given that we can differentiate the two original functions). We will then discuss the derivates of the six standard trigonometric functions. Of these, the most important are sine and cosine; the derivatives of all the other standard trigonometric follow easily from these.

The reference for today is Stewart §3.2 (for products and quotients) and §3.3 (trigonometric functions).

2 The product rule

The product rule has a fairly simple form.

$$\boxed{\frac{d}{dx} f(x)g(x) = \frac{df}{dx} g(x) + f(x)\frac{dg}{dx}}$$

There are many other ways to write it. Another favorite is this rather concise alternative.

$$\boxed{(fg)' = f'g + fg'}$$

The way I usually remember the product rule is with the following explanation: the change in $f(x)g(x)$ is the sum of the change due to $f(x)$ and the change due to $g(x)$. If it were just $f(x)$ changing, then the derivative would just be $f'(x)g(x)$ ($g(x)$ would behave like a constant), while if it were just $g(x)$ changing it would be the reverse. The total change of $f(x)g(x)$ simply has both these contributions.

A nice illustration of the product rule (and a good way to confirm that it makes sense) is to see what is says about the function x^2. This is just the same thing as $x \cdot x$, so the product rule easily computes that $\frac{d}{dx}x^2 = \frac{dx}{dx}x + x\frac{dx}{dx} = x + x = 2x$. To my mind, this is the actually the best way to understand why the derivative of x^2 is $2x$ (rather than, say, x, or $3x$).

Here are a couple more examples.

Example 2.1. Consider $f(x) = xe^x$. Then $f'(x) = \frac{dx}{dx}e^x + x\frac{d}{dx}e^x = e^x + xe^x$, which could also be written $(x+1)e^x$. In fact, if you continue taking derivatives, a nice pattern emerges: $f''(x) = \frac{d}{dx}(x+1)e^x + (x+1)\frac{d}{dx}e^x = e^x + (x+1)e^x = (x+2)e^x$, and the same sort of reasoning shows that $f'''(x) = (x+3)e^x$ and so forth.

Example 2.2. Consider $f(x) = (x^7+1)(x^9+1)$. There are two ways to differentiate this function; both are easy in different ways (and it is often smart to try to do both methods, as a way of checking your work). One method is the product rule: $f'(x) = (7x^6)(x^9+1)+(x^7+1)(9x^8) = 7x^{15}+7x^6+9x^{15}+9x^8 = 16x^{15}+9x^8+7x^6$. Another is to begin by expanding the function out to $f(x) = x^{16} + x^9 + x^7 + 1$, and differentiate this with the power rule to obtain the same answer.

Example 2.3. Consider $f(x) = e^{2x}$. We will shortly see that the "chain rule" gives an easy way to differentiate this function, but at the moment we can easily differentiate it with the product rule also: $f(x) = e^x \cdot e^x$, so $f'(x) = \frac{de^x}{dx}e^x + e^x\frac{de^x}{dx} = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$.

3 The quotient rule

The quotient rule has a somewhat more nauseating form than the product rule. The usual expression is the following (both of these are equivalent, just using different notation for the derivative).

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{g(x)^2}}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

,

3.1 Alternative forms and derivations

Personally, I find it much easier, instead of trying to remember these sorts of ugly formulas, to just remember one special case, which is easy to remember since it closely resembles $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

From here, the quotient rule follows just by applying the product rule.

$$\left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} - f \cdot \frac{g'}{g^2}$$

In fact, the equation above is the form in which I usually prefer to apply the quotient rule myself.

There is also a fairly quick derivation of the quotient rule from the product rule, as follows.

$$\begin{aligned}
f &= g \cdot \frac{f}{g} \\
\Rightarrow f' &= \left(g \cdot \frac{f}{g}\right)' \\
&= g'\frac{f}{g} + g\left(\frac{f}{g}\right)' \\
\Rightarrow f' - g'\frac{f}{g} &= g\left(\frac{f}{g}\right)'
\end{aligned}$$

Dividing both sides of this equation by g (and doing some mild simplifying) now gives the quotient rule.

3.2 Examples

One of the main uses of the quotient rule is to differentiate rational functions. For example:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x+1}\right) &= \frac{(\frac{d}{dx}1)(x+1) - 1\frac{d}{dx}(x+1)}{(x+1)^2} \\
&= -\frac{1}{(x+1)^2} \\
\frac{d}{dx}\left(\frac{2x+3}{x^2+1}\right) &= \frac{\left[\frac{d}{dx}(2x+3)\right](x^2+1) - (2x+3)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{2(x^2+1) - (2x+3)(2x)}{(x^2+1)^2} \\
&= \frac{-2x^2 - 6x + 2}{(x^2+1)^2}
\end{aligned}$$

It is also often necessary to combine the quotient rule with the product rule, as in the following example, where the derivative of xe^x appears in the expression and must be evaluated using the product rule.

$$\begin{aligned}
f(x) &= \frac{xe^x}{x^2 - x + 1} \\
f'(x) &= \frac{\left[\frac{d}{dx}(xe^x)\right](x^2 - x + 1) - xe^x\frac{d}{dx}(x^2 - x + 1)}{(x^2 - x + 1)^2} \\
&= \frac{(e^x + xe^x)(x^2 - x + 1) - xe^x(2x - 1)}{(x^2 - x + 1)^2} \\
&= e^x \cdot \frac{(1+x)(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} \\
&= e^x \cdot \frac{x^3 + 1 - 2x^2 + x}{(x^2 - x + 1)^2} \\
&= e^x \cdot \frac{x^3 - 2x^2 + x + 1}{(x^2 - x + 1)^2}
\end{aligned}$$

4 Sine and cosine

We've seen how to differentiate polynomials and exponential functions. The other main missing piece in the catalog of elementary functions that can be easily differentiated are the trigonometric functions. The two basic facts are the following.

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x
\end{aligned}$$

So these two basic functions are closely linked to each other; the main confusing thing is to remember which one obtains a negative sign when it is differentiated. The easiest way to get straight on this is just to think about where the functions are increasing and decreasing. The graph of sine is initially increasing, so its derivative at 0 had better by positive; thus is must be $\cos x$ and not $-\cos x$. Similarly, the graph of $\cos x$ starts at a local maximum, so its derivative but change from positive to negative around $x = 0$; this is the opposite of what $\sin x$ does, so the derivative of $\cos x$ must be $-\sin x$ and not $\sin x$.

Observation. Both sine and cosine have the very special property that they are the *negative* of their second derivative, i.e. $(\sin x)'' = -\sin x$ and $(\cos x)'' = -\cos x$. In physical terms, each curve is always accelerating back towards the x axis at a rate given by its distance from the x axis. This is the reason that these functions arise so much in physical problems: any system with "feedback" that pulls it back towards equilibrium (e.g. a weight on a spring, or a swaying bridge) is governed by some equations that ultimately give rise to functions that are built up from sine and cosine.

Note. The fact that the derivatives of sine and cosine have such a nice form in terms of each other is the principle reason why radians, rather than degrees, are always used when doing trigonometry (at least when any techniques from calculus are begin used). It is analogous to using the metric system in chemistry: just like the metric system makes unit conversions less error-prone, using radians makes taking derivatives less error-prone. This is exactly analogous to using e^x rather than any other exponential function[1].

4.1 The derivation

The usual derivation of the derivatives of sine and cosine uses the following standard trigonometric identities (the word "identity" means a formula which hold for all values of the input). I've put everything involving h in blue to make it stand out.

$$\begin{aligned} \sin(x+h) &= \sin x \cos h + \cos x \sin h \\ \cos(x+h) &= \cos x \cos h - \sin x \sin h \end{aligned}$$

The basic facts that allow us to compute the derivatives of sine and cosine are the following *linear approximations* to $\sin x$ and $\cos x$: for x for *close to* 0,

$$\begin{aligned} \sin x &\approx x \\ \cos x &\approx 1 \end{aligned}$$

The first of these was an identity we discussed in the lecture on linear approximation. The second follows because $\cos x$ has a local maximum at $x = 0$, hence a horizontal tangent line.

Applying these linear approximations and the identities above, we obtain the following fact: if h is a very small number (very close to 0), then:

$$\begin{aligned} \sin(x+h) &\approx \sin x \cdot 1 + \cos x \cdot h \\ \cos(x+h) &\approx \cos x \cdot 1 - \sin x \cdot h \end{aligned}$$

From these approximations, we see that $\sin x$ increases at a rate of $\cos x$ (as h increases), while $\cos x$ increases at a rate of $-\sin x$.

The more formal version of what I have just said is to first invoke the following two limits (which are both computed by some analysis using the squeeze theorem, which we will not describe in detail). Both are visually plausible if you draw the graphs of sine and cosine; they say that the derivative of sine at 0 is 1, and that the derivative of cosine at 0 is 0.

[1] If you happen to have seen the formula, in terms of complex numbers, $e^{ix} = \cos x + i \sin x$, you will realize that choosing e and choosing radians are really the exact same choice, if you take a slightly broader point of view.

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$
$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

Then the derivative of $\sin x$ can be computed as follows.

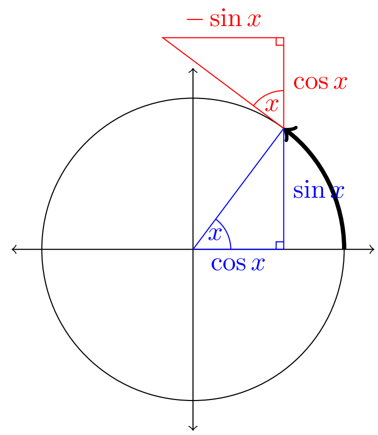
$$\begin{aligned}
(\sin x)' &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right) \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}$$

The derivative of $\cos x$ can be formally commuted in a totally analogous way.

$$\begin{aligned}
(\cos x)' &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\
&= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x
\end{aligned}$$

4.2 A physical interpretation

Skip this subsection if you don't particularly like physics. But I find the following picture to be the clearest explanation for why the derivatives of sine and cosine are what they are.



In this picture, you should imagine the curved arc as a planet orbiting the origin (1 unit away) in a perfect circle, traveling at speed exactly 1. Then the velocity of this orbiting planet will point in a direction tangent to the circle, and will have magnitude 1. Then you can determine the x and y coordinates of velocity by drawing the red triangle shown. It is congruent to the blue triangle, but rotated 90 degrees. Then the derivatives of sine and cosine can be seen by immediate visual inspection in this picture.

4.3 Some examples

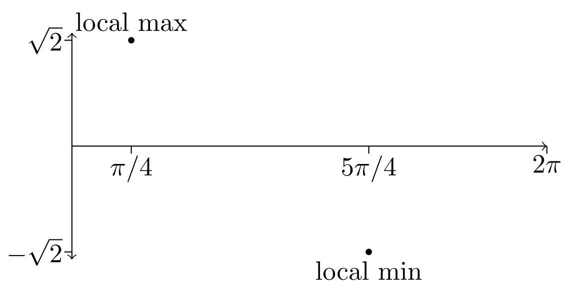
Example 4.1. Sketch the function $f(x) = \sin x + \cos x$.
Solution. This function is periodic with period 2π (since both sine and cosine are), so let's first restrict to the interval $[0, 2\pi]$.

We can easily compute the first two derivatives.

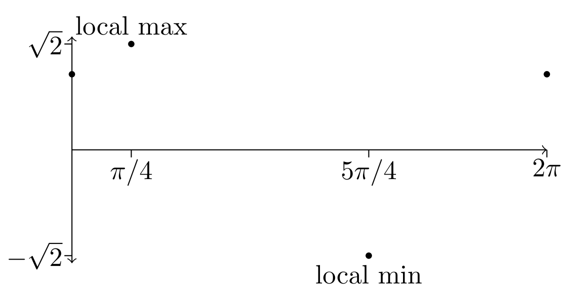
$$\begin{aligned} f'(x) &= \cos x - \sin x \\ f''(x) &= -\sin x - \cos x \end{aligned}$$

From the first derivative, we can identify all of the critical points: these are the points where $\cos x - \sin x = 0$, i.e. $\cos x = \sin x$, i.e. $\tan x = 1$. This occurs precisely at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. We can classify these critical points using the second derivative: $f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$, and $f''(\frac{5\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$. So there is a local maximum at $\frac{\pi}{4}$ and a local minimum at $\frac{5\pi}{4}$. We can compute that these values are $f(\frac{\pi}{4}) = \sqrt{2}$ and $f(\frac{5\pi}{4}) = -\sqrt{2}$.

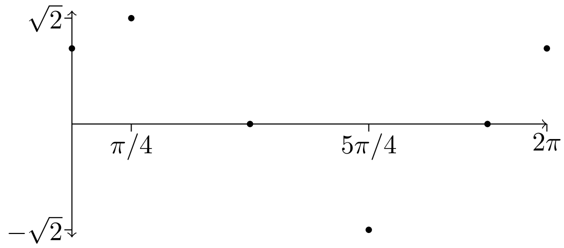
So far, we've found the following information about the graph.



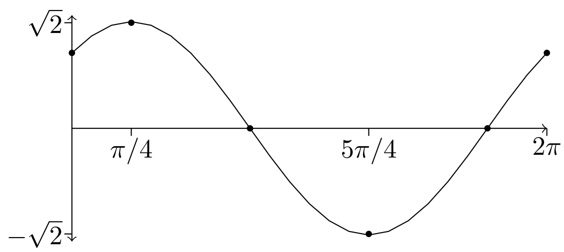
Let's fill out this picture a bit more. First, we could evaluate the function at 0 and 2π to find that $f(0) = f(2\pi) = 1$.



Next, we could find the x-intercepts of the function. These are the values of x where $\sin x = -\cos x$, which are precisely $3\pi/4$ and $7\pi/4$. So plot these points also.



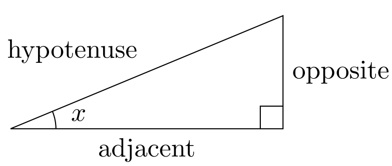
Finally, notice that $f''(x)$ is precisely $-f(x)$, so the graph will be **concave down where it is positive** and **concave up where it is negative**. It will have inflection points precisely where it crosses the x-axis. Using this information, we can make a pretty good sketch of the graph.



Note. If you are very skilled with trigonometry, you might also just notice that $\sin x + \cos x$ is precisely the same as $\sqrt{2}\sin(x + \frac{\pi}{4})$. This would allow you to sketch this particular graph quite readily.

5 The six standard trigonometric functions

As you may have seen in your precalculus, there are six functions that usually make up the "standard trigonometric functions." As follows.



Function name	Notation	Definition
sine	$\sin x$	opposite / hypotenuse
tangent	$\tan x$	opposite / adjacent
secant	$\sec x$	hypotenuse / adjacent

Function name	Notation	Definition
cosine	$\cos x$	adjacent / hypotenuse
cotangent	$\cot x$	adjacent / opposite
cosecant	$\csc x$	hypotenuse / opposite

The nomenclature here is a bit of a nightmare, I'm afraid. All these terminology is couple hundred years old, and it one of those vestigial organs that we cannot seem to excise from common usage. However, this is the predominant nomenclature for these functions so it is worth reviewing them.

Here's one feature that makes all this slightly easier: these six functions are arranged into four "dual" pairs: sine and cosine; tangent and cotangent; secant and cosecant. Each function is related to its "dual" in a simple way: just swap "adjacent" and "opposite" wherever you see them[2].

All six are derived very directly from sine and cosine, as follows.

[2] A more mathematical way to say this is: replace x with $\pi/2 - x$ (so $\cos(x) = \sin(\frac{\pi}{2} - x)$, $\cot x = \tan(\frac{\pi}{2} - x)$, and $\csc x = \sec(\frac{\pi}{2} - x)$.

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\csc x &= \frac{1}{\sin x}
\end{aligned}$$

As a result, it is straightforward to differentiate all of these functions by use of the quotient rule and the derivatives of sine and cosine. It is worth working these computations yourself, as an exercise in the quotient rule. The computations are shown below.

I do not suggest that you memorize these. Instead, I suggest that you practice deriving them, so that you can always do it in a couple seconds when needed. This is both good practice with the quotient rule (which will pay dividends in more complex computations), and also will help you avoid overburdening your brain with arbitrary formulas.

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\cot x &= \frac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\sec x &= \frac{0 - 1\cdot(\cos x)'}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\csc x &= \frac{0 - 1\cdot(\sin x)'}{\sin^2 x} \\
&= -\frac{\cos x}{\sin^2 x} \\
&= -\frac{1}{\sin x}\frac{\cos x}{\sin x} \\
&= -\csc x \cot x
\end{aligned}$$

Note that there are generally many ways to write these derivatives (for example, the derivative of $\tan x$ could be written either $\frac{1}{\cos^2 x}$ or $\sec^2 x$). The convention is generally to write functions without denominators if possible; this entails replacing $\frac{1}{\cos x}$ with $\sec x$ where possible, for example.

By the way, here's another pattern that sometimes saves a little effort: if you want to derivative of a "dual" function, take the "negative dual" of the original function. So since $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$. Similarly, since $\frac{d}{dx}\tan x = \sec^2 x$, we can conclude that $\frac{d}{dx}\cot x = -\csc^2(x)$ (because cosecant is dual to secant, then you must add a negative sign also).