

Lecture 10: Linear Approximation

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1 Introduction

Now that we have spent some time discussing what exactly the derivative is, and how it is defined using limits, we will discuss a couple of the main applications of the derivative. For now, we'll limit ourselves to fairly simple functions: those that can be differentiated using just the rules we saw in lecture 9 (the power rule and the derivative of e^x). Later, we'll cover more versatile differentiation shortcuts and return to these applications again.

The first application we consider is called linear approximation. The basic idea of linear approximation is “local linearity:” this idea says that a tangent line to a function really lies quite close to the function, at least near the point of tangency. Therefore it can serve as a very easily computed and conceptually simple approximation of the original function.

The reference for today is Stewart §3.9.

2 Preliminaries

Recall the following three facts (the first two were discussed in lecture 9; the last one was discussed in homework problem 3A – 5(ii)). These will be used extensively today.

- $\frac{d}{dx}x^n = nx^{n-1}$, where n is *any* constant, whether positive or negative, whole number or not.
- $\frac{d}{dx}e^x = e^x$.
- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ (note that this is *only true if x is the number of radians*, not degrees).

The third of these facts amounts to saying that if $f(x) = \sin x$, then $f'(0) = 1$. Later we will see what the derivative of $\sin x$ is everywhere; for now it will suffice just to know its value at 0.

3 Linear approximation

The simplest of all functions are *linear* functions – those functions whose graphs are straight lines. The idea of *linear approximation* is that, when perfect accuracy is not needed, it is often very useful to approximate a more complicated function by a linear function.

Definition 3.1. The *linear approximation* of a function $f(x)$ around a value $x = c$ is the following linear function. Remember: c is a constant that you have chosen, so this is just a function of x .

$$L(x) = f(c) + f'(c) \cdot (x - c)$$

The graph of this function is precisely the same as the tangent line to the curve $y = f(x)$. The function $L(x)$ is the unique linear function that satisfies $L(c) = f(c)$ and $L'(c) = f'(c)$. Sometimes we say that it “matches the function $f(x)$ to first order” for this reason.

Example 3.2. Physicists often use the approximation $\sin x \approx x$. This approximation is particularly useful in optics, where an important relationship (Snell's law) becomes vastly simplified if you replace $\sin x$ with x wherever it appears. This simplification allow relatively easy computation of how curved a lens should be to achieve a certain magnification (for example).

This approximation is nothing but the linear approximation of the function $f(x) = \sin x$ around the value $c = 0$. You can see this as follows:

$$\begin{aligned} f(0) &= \sin 0 \\ &= 0 \\ f'(0) &= \lim_{h \rightarrow 0} \frac{\sin(h) - \sin 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= 1 \\ L(x) &= f(0) + f'(0) \cdot (x - 0) \\ &= x \end{aligned}$$

The last inequality in computing $f'(0)$ is simply the fact we mentioned in the preliminaries: that the ratio of $\sin x$ to x goes to 1 as x goes to 0.

In the case of $\sin x \approx x$, the linear approximation is useful because it vastly simplifies other calculations. Another use for linear approximation is to calculate unknown values of a function given known values.

The linear approximation is useful if you can calculate $f(c)$ and $f'(c)$ exactly (or estimate them well), but you don't know how to calculate nearby values of the function. This is sometimes useful for quick mental approximations; it is also a basic technique that underlies how computers actually go about computing values of functions like square roots.

Example 3.3. Estimate $\sqrt{26}$ using a linear approximation.

Solution. Let $f(x) = \sqrt{x}$. Then we wish to approximation $f(26)$. To do this, look for a nearby input to this function that we *can* compute easily: $f(25) = \sqrt{25} = 5$. To do a linear approximation, we also need to know the value of the derivative of f at 25. But we can do this, too, using the power rule for differentiation.

$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{1/2} \\ f'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-1/2} \\ &= \frac{1}{2\sqrt{x}} \\ \Rightarrow f'(25) &= \frac{1}{2 \cdot 5} \\ &= \frac{1}{10} \end{aligned}$$

Therefore the linear approximation of \sqrt{x} around $x = c$ is the following function.

$$\begin{aligned} L(x) &= f(25) + f'(25)(x - 25) \\ &= 5 + \frac{1}{10}(x - 25) \end{aligned}$$

To approximate $\sqrt{26}$, just plug $x = 26$ into this function to obtain $\sqrt{26} \approx 5 + \frac{1}{10}(26 - 25) = 5 + \frac{1}{10} = 5.1$. Indeed, 5.1 is very close to the actual square root of 26; its square is 26.01, in fact.

4 Examples

We will now use linear approximation around a suitable value $x = c$ to estimate the following numbers.

1. $e^{0.017}$
2. $\frac{12}{1.02}$
3. $\sqrt[3]{9}$
4. $\sqrt{50}$
5. $\sin(1.8^\circ)$ Remember: 1.8° (that is, 1.8 degrees) is not the same thing as 1.8 radians.
6. $f(5)$, where $f(x)$ is the function $f(x) = \frac{x + 200}{\sqrt{x}}$.

The steps for each example are the same: identify a function $f(x)$ such that the desired quantity is some value, but there is a nearby value that is easy to compute. Then determine the linear approximation function and substitute the desired value of x . These steps will be shown in parallel for the six examples.

Desired quantity	$e^{0.017}$	$\frac{12}{1.02}$	$\sqrt[3]{9}$	$\sqrt{50}$	$\sin(1.8^\circ)$	$\frac{5+200}{\sqrt{5}}$
$f(x)$	e^x	$\frac{12}{x}$ $= 12x^{-1}$	$\sqrt[3]{x}$ $= x^{1/3}$	\sqrt{x} $= x^{1/2}$	$\sin x$	$\frac{x+200}{\sqrt{x}}$ $= x^{1/2} + 200x^{-1/2}$
$f'(x)$	e^x	$-12x^{-2}$	$\frac{1}{3}x^{-2/3}$	$\frac{1}{2}x^{-1/2}$?	$\frac{1}{2}x^{-1/2} - 100x^{-3/2}$
center c	0	1	8	49	0	4
$f(c)$	1	12	2	7	0	102
$f'(c)$	1	-12	$\frac{1}{12}$	$\frac{1}{14}$	1	$\frac{1}{4} - \frac{100}{8}$ $= -\frac{49}{4}$
$L(x)$	$1 + x$	$12 - 12(x - 1)$	$2 + \frac{1}{12}(x - 8)$	$7 + \frac{1}{14}(x - 49)$	x	$102 - \frac{49}{4}(x - 4)$

Now that we have a linear approximation function suitable to approximate each desired quantity, we can approximate them as follows. Note that the function $L(x)$ used in each item below is different; it is the function found in the table above. I've also noted the "true" value, to enough decimal places to see where it starts to differ from the the value computed by linear approximation.

1. $e^{0.017} \approx L(0.017) = 1 + 0.017 = 1.017$. The true is 1.0174 (to four decimal places).
2. $\frac{12}{1.02} \approx L(1.02) = 12 - 12 \cdot 0.02 = 11.76$. The true value is 11.765 (to three decimal places).
3. $\sqrt[3]{9} \approx L(9) = 2 + \frac{1}{12} \cdot 1 = 2\frac{1}{12} \approx 2.083$. The true value is 2.080 (to three decimal places).
4. $\sqrt{50} \approx 7 + \frac{1}{14} \cdot 1 \approx 7.0714$. The true value is 7.0710 (to four decimal places).
5. To compute $\sin(1.8^\circ)$, we should first convert 1.8° to radians, since otherwise our value for the derivative of $\sin x$ at 0 (and thus our linear approximation) is wrong. There are 2π radians in 360° , so 1.8 degrees is equal to $1.8 \cdot \frac{2\pi}{360} = \frac{\pi}{100}$ radians. To approximate $\sin 1.8^\circ$, use the linear approximation $\sin x \approx L(x) = x$ (where x is in radians) to obtain $\sin(1.8^\circ) \approx \frac{\pi}{100} \approx 0.031415$. The true value is 0.031410 (to six decimal places).
6. $f(5) \approx L(5) = 102 - \frac{49}{4} \cdot 1 = 89\frac{3}{4} = 89.75$. The true value is 91.7 (to one decimal place).

5 Over or under estimate?

Recall that one way to describe a concave up function is that it lies above its tangent line. So the concavity of a function can tell you whether the linear approximation will be an overestimate or an underestimate.

1. If $f(x)$ is concave **up** in some interval around $x = c$, then $L(x)$ **underestimates** in this interval.
2. If $f(x)$ is concave **down** in some interval around $x = c$, then $L(x)$ **overestimates** in this interval.

Remember that an easy way to determine concavity is to evaluate the second derivative. For example, consider the six examples from the previous section.

1. If $f(x) = e^x$, then $f''(x) = e^x$, which is always positive, so $f(x)$ is concave up. So $L(0.017) = 1.017$ is an underestimate of $e^{0.017}$.
2. If $f(x) = 12/x$, then $f''(x) = 24x^{-3}$, which is positive for all positive x , so $f(x)$ is concave up for positive x . So $L(1.02) = 11.76$ is an overestimate of $12/1.02$.
3. If $f(x) = \sqrt[3]{x}$, then $f''(x) = -\frac{2}{9}x^{-5/3}$, which negative for positive x , so $f(x)$ is concave down for positive x . So $L(9) = 2\frac{1}{12}$ is an overestimate of $\sqrt[3]{9}$.
4. If $f(x) = \sqrt{x}$, then $f''(x) = -\frac{1}{4}x^{-3/2}$, which is negative for positive x , so $f(x)$ is concave down for positive x . So $L(50) = 7\frac{1}{14}$ is an overestimate of $\sqrt{50}$.
5. If $f(x) = \sin x$, then we don't yet know how to compute $f''(x)$. So we can't yet conclude anything about whether $\sin x \approx x$ gives an over or under estimate.
6. If $f(x) = \frac{x+200}{\sqrt{x}}$ then $f''(x) = -\frac{1}{4}x^{-3/2} + 150x^{-5/2}$. At $c = 4$, $f''(c) = -\frac{1}{4}\frac{1}{8} + 150\frac{1}{32} = -\frac{1}{32} + \frac{150}{32} = \frac{149}{32}$. So around $c = 4$, the linear approximation is an underestimate (the function is concave up). Indeed, $L(5) = 89.75$ is an underestimate of $f(5)$.

6 Appendix: A limit expression for the number e

As we've said, basing all exponential functions on the natural exponential function $f(x) = e^x$ is like using the metric system: it makes a lot of conversion easier. One way in which e^x is fairly natural is that it has a very simple linear approximation around $x = 0$: since $f(0) = f'(0) = 1$, the linear approximation is $e^x \approx 1 + x$.

This linear approximation (like all linear approximations) gets better and better the closer x is to 0. This fact turns out to provide one way of writing down a precise definition of Euler's number e . Indeed, we have the following estimate for the n th roots of e , for all n .

$$\sqrt[n]{e} = e^{1/n} \approx 1 + \frac{1}{n}$$

Taking n th powers of both sides, this gives

$$e \approx \left(1 + \frac{1}{n}\right)^n$$

Note that since e^x is concave up, we know that this is always an underestimate.

This approximation gets better and better the larger n becomes. So the "ideal" value, as n grows arbitrarily large, is just Euler's number e . The precise statement, which you may have seen in precalculus, is the following.

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

There are many other definitions of e , and of course they are all equivalent; I mention this one now since it comes from the current topic of linear approximation in a natural way.