Worksheet for 21 November

Part I (go on to Part II if you finish)

① $\int_2^9 \frac{1}{\sqrt{x+7}}\,dx = \int_9^{16} \frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_9^{16}$

$u = x+7$
$du = dx$

$= 2\cdot 4 - 2\cdot 3 = \boxed{2}$

② $\int_0^4 x\cdot e^{x^2}dx = \int_0^{16} \frac{1}{2}e^u du = \left[\frac{1}{2}e^u\right]_0^{16}$

$u = x^2$
$du = 2x\,dx$

$= \boxed{\frac{1}{2}(e^{16}-1)}$

③ $\int_{\pi/6}^{\pi/3} \sin^3 x\cos x\,dx = \int_{1/2}^{\sqrt{3}/2} u^3 du = \left[\frac{1}{4}u^4\right]_{1/2}^{\sqrt{3}/2}$

$u = \sin x$
$du = \cos x\,dx$

$= \frac{1}{4}\cdot\frac{9}{16} - \frac{1}{4}\cdot\frac{1}{16} = \frac{8}{64} = \boxed{\frac{1}{8}}$

④ $\int_0^{\pi/4} \tan x\,dx$

$= \int_0^{\pi/4} \frac{\sin x}{\cos x}dx = \int_1^{\sqrt{2}/2} \frac{-du}{u} = \left[-\ln|u|\right]_1^{\sqrt{2}/2}$

$u = \cos x$
$du = -\sin x\,dx$

$= -\ln\frac{\sqrt{2}}{2} + \ln 1$

$= -\ln\frac{\sqrt{2}}{2} = \ln\frac{2}{\sqrt{2}} = \ln\sqrt{2}$

$= \boxed{\frac{1}{2}\ln 2}$

Part II

① $\int_0^{\pi} \frac{e^x + \cos x}{e^x + \sin x}\,dx = \int_1^{e^{\pi}} \frac{du}{u} = \left[\ln|u|\right]_1^{e^{\pi}}$

$u = e^x + \sin x$ $\qquad = \boxed{\pi}$

$du = (e^x + \cos x)\,dx$

② $\int x\sqrt{x+1}\,dx = \int x\sqrt{u}\,du$

$u = x+1$ $\qquad = \int (u-1)\sqrt{u}\,du$

$du = dx$ $\qquad = \int (u^{3/2} - u^{1/2})\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C}$

③ $\int x^3(x^2+1)^{21}\,dx = \int x^3 \cdot u^{21} \cdot \frac{du}{2x} = \int \frac{1}{2}x^2 \cdot u^{21}\,du$

$u = x^2+1$

$du = 2x\,dx$ $\qquad = \int \frac{1}{2}(u-1)u^{21}\,du = \int \frac{1}{2}(u^{22} - u^{21})\,du$

$\Rightarrow dx = \frac{du}{2x}$ $\qquad = \left[\frac{1}{2}\cdot\frac{1}{23}u^{23} - \frac{1}{2}\cdot\frac{1}{22}\cdot u^{22}\right]$

$= \boxed{\frac{1}{46}(x^2+1)^{23} - \frac{1}{44}(x^2+1)^{22}}$

④ $\int_{\ln 2}^{\ln 3} \frac{e^{2x}}{e^x+1}\,dx = \int_3^4 \frac{e^{2x}}{u}\cdot\frac{du}{e^x}$

$u = e^x + 1$

$du = e^x dx$

$= \int_3^4 \frac{e^x}{u}\,du = \int_3^4 \frac{u-1}{u}\,du = \int_3^4 \left(1 - \frac{1}{u}\right)du$

$= \left[u - \ln|u|\right]_3^4 = (4 - \ln 4) - (3 - \ln 3)$

$= \boxed{1 - \ln 4 + \ln 3}$

⑤ $\int \frac{1}{x[(\ln x)^2+4\ln x+4]}\,dx = \int \frac{1}{u^2+4u+4}\,du$

$u=\ln x$
$du=\frac{1}{x}dx$

$= \int \frac{1}{(u+2)^2}du = \int \frac{1}{v^2}dv = -\frac{1}{v}+C$

$v=u+2$
$du=dv$

$= -\frac{1}{u+2}+C = \boxed{-\frac{1}{\ln x+2}+C}$

⑥ $\int \frac{1}{4+x^2}dx$

$= \int \frac{1}{4}\cdot\frac{1}{1+(x/2)^2}dx = \int \frac{1}{4}\cdot\frac{1}{1+u^2}\cdot 2du = \frac{1}{2}\int \frac{1}{1+u^2}du$

$u=x/2$
$du=\frac{1}{2}dx$

$= \frac{1}{2}\tan^{-1}u+C = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C}$

⑦ $\int \cos^3 x \sin^2 x\,dx = \int \cos^2 x\cdot u^2\,du$

$u=\sin x$
$du=\cos x\,dx$

$= \int (1-u^2)u^2\,du$

$= \int (u^2-u^4)\,du$

$= \frac{1}{3}u^3-\frac{1}{5}u^5\,du$

$= \boxed{\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C}$

Challenges (for the intrepid)

(8) $\int \cos(\sqrt{x})\,dx = \int \cos(u)\cdot 2\sqrt{x}\,du$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$dx = 2\sqrt{x}\cdot du$

$= \int 2u\cos(u)\,du$

$= 2u\sin u + 2\cos u + C$ (if you're a good guesser; or using int. by parts, to be covered next time).

$= \boxed{2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C}$

(9) $\int \sec^4 x\,dx$

$= \int (1+\tan^2 x)\cdot \sec^2 x\,dx = \int (1+u^2)\,du$

$u = \tan x$

$du = \sec^2 x\,dx$

$= u + \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{3}\tan^3 x + \tan x + C}$

(10) $\int \frac{(1+x^2)e^x}{x^2\cdot e^{1/x}}\,dx$

$= \int \left(\frac{1}{x^2}+1\right)\cdot e^{x-1/x}\,dx = \int e^u\,du$

$u = x - 1/x$

$du = 1 + 1/x^2$

$= e^u + C$

$= \boxed{e^{x-1/x} + C}$