Review problems

Part I: fast recall (no explanations needed)

Differentiate:

1. \( \frac{d}{dx} x^2 \) = 2x
2. \( \frac{d}{dx} \sqrt{x} \) = \( \frac{1}{2\sqrt{x}} \)
3. \( \frac{d}{dx} \frac{1}{x} \) = \(-\frac{1}{x^2}\)
4. \( \frac{d}{dx} e^x \) = e^x
5. \( \frac{d}{dx} \ln(x) \) = \( \frac{1}{x} \)
6. \( \frac{d}{dx} \cos(x) \) = \(-\sin(x)\)
7. \( \frac{d}{dx} \tan(x) \) = \( \sec^2(x) \)
8. \( \frac{d}{dx} \frac{1}{\cos(x)} \) = \( \frac{1}{\cos^2(x)} \)
9. \( \frac{d}{dx} \frac{1}{\sin(x)} \) = \(-\frac{\cos(x)}{\sin^2(x)} \)
10. \( \frac{d}{dx} x \cdot e^x \) = \( x \cdot e^x + e^x \)

Estimate \( \sin(0.02) \) with linear approximation.

\( \sin(x) \approx x \) near \( x = 0 \)

\[ \Rightarrow \sin(0.02) \approx 0.02 \]

Optimization

3. Find the minimum of \( f(x) = e^x - x \) on \( (-\infty, \infty) \).

\[ f'(x) = e^x - 1 \quad \text{and} \quad f''(x) = e^x \]

so \( x = 0 \) is the absolute minimum: \( \min = (0, 1) \)

4. Find the maximum and minimum of \( f(x) = x^2 - 3x \) on \([0, 2]\).

\[ f'(x) = 2x - 3 = 3(x - 1) \quad \text{and} \quad f''(x) = 2 \]

so \( x = 1 \) is the maximum on \([0, 2]\) and \( x = 0, 2 \) are end-points.

Graphing

5. Sketch the graph of \( f(x) = x^3 - 3x \).

Evaluate

6. \( \lim_{x \to \infty} \frac{e^{x+1}}{e^x - 1} \) and \( \lim_{x \to 0} \frac{e^{x+1}}{e^x - 1} \).

\[ \frac{e^{x+1}}{e^x - 1} = \frac{e^x \cdot e}{e^x - 1} = \frac{e}{1 - \frac{1}{e^x}} = \frac{1 + e^{-x}}{1 - e^{-x}} \]

related note 7

Suppose \( f, g \) are functions such that \( f(t)g(t) = t \) for all \( t \),

\( f(0) = 1 \) and \( g(0) = 0 \) and \( f'(0) = 1 \). Find \( g'(0) \).

\[ f'g + g'f = 1 \quad \text{and} \quad g'(0) = \frac{1 - 1 \cdot 0}{0} \]

\[ g'f = 1 - f'g \quad \text{and} \quad g' = \frac{1 - f'g}{f} \]

\[ g'(0) = \frac{1}{1/7} = 7 \]
Part II: longer problems.

8. (Differentiation) Find derivatives:
   a) \( \frac{d}{dx} \ln(\ln(\ln(x))) = \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x} \)

   b) \( \frac{e^x \sqrt{\sin x}}{(x+1)^2 (x-2)} \)

   \( \ln f(x) = x + \frac{1}{2} \ln(\sin x) - \ln(\ln(x+1)) - \ln(\ln(x-2)) \)

   \( f'(x) = 1 + \frac{1}{2} \cdot \frac{\cos x}{\sin x} - \frac{1}{x+1} - \frac{1}{x-2} \)

   \( f'(x) = \frac{e^x \sqrt{\sin x}}{(x+1)^2 (x-2)} \cdot \left[ 1 + \frac{1}{2} \cdot \frac{\cos x}{\sin x} - \frac{1}{x+1} - \frac{1}{x-2} \right] \)

   c) sec \(-x\), for \(x > 0\)

   \( f(x) = \sec x \)

   \( f'(x) = \frac{1}{\sec(\sec^{-1} x)} \cdot \tan(\sec^{-1} x) \)

   \( f'(x) = \frac{1}{x} \cdot \sqrt{x^2 - 1} \)

   Noting for \(x > 0\),

   \( \tan(\sec^{-1} x) = \sqrt{x^2 - 1} \)

   so \( f'(x) = -\frac{x}{\sqrt{x^2 - 1}} \)

   in general \( f'(x) = \frac{1}{x} \cdot \sqrt{x^2 - 1} \)

   \( f(x) = |x| \cdot \sqrt{x^2 - 1} \)

   (not needed here)

   d) \( e^x \cdot \ln(7x+1) \)

   Product rule:

   \( f'(x) = \left(2x e^x\right) \ln(7x+1) + e^x \cdot \frac{7}{7x+1} \)

9. (Optimization)

   a) Find the maximum area of a right triangle with hypotenuse of length 1.

   The area is \( \frac{1}{2} ab \)

   \( b = \sqrt{1-a^2} \)

   \( f(a) = \frac{1}{2} a \sqrt{1-a^2} \text{ on } [0,1] \)

   \( f'(a) = \frac{1}{2} \left( \sqrt{1-a^2} - \frac{a^2}{\sqrt{1-a^2}} \right) \)

   => \( 1-a^2 = a^2 \)

   \( \Rightarrow a = \pm \sqrt{\frac{1}{2}} \)

   \( f(0) = 0, f(\sqrt{\frac{1}{2}}) = \frac{1}{2}, f(1) = 0 \)

   => \( \max = \frac{1}{2} \)

   b) Find the maximum perimeter of a right triangle with hypotenuse of length 1.

   \( a^2 + b^2 = 1 \)

   \( b = \sqrt{1-a^2} \)

   \( f(a) = a + \sqrt{1-a^2} + 1 \)

   \( f'(a) = 1 - \frac{a}{\sqrt{1-a^2}} \)

   \( f'(a) = 0 \) => \( a = \frac{1}{\sqrt{2}} \)

   \( f(0) = 2, f(\frac{1}{\sqrt{2}}) = \sqrt{2} + 1 \)

   \( f(1) = 2 \)

   So \( \max = \sqrt{2} + 1 \)
10) (optimization)

Find the minimum perimeter of a slice of pie of area 50 cm².

\[ \text{Area} = \frac{1}{2} \theta r^2 = 50 \]

Perimeter = 2r + r \theta

\[ = \theta = \frac{50}{\frac{1}{2}r^2} = 100/r^2 \]

\[ P(r) = 2r + r \cdot \frac{100}{r} \]

\[ = 2r + \frac{100}{r} \]

\[ P'(r) = 2 - 100/r^2 \]

\[ P''(r) = 200/r^3 \]

want to minimize \( P(r) \) on \( r > 0 \).

concave up \( \Rightarrow \) local min will be absolute.

\[ P'(r) = 0 \quad \Rightarrow \quad \frac{100}{r^2} = 2 \]

\[ \Rightarrow \quad r = \sqrt{50} \]

so minimum is \( P(\sqrt{50}) \)

\[ = 2\sqrt{50} + \frac{100}{\sqrt{50}} \]

\[ = \frac{20\sqrt{2} + 10\sqrt{2}}{2} \]

11) (linear approximation)

a) Approximate \( \tan^{-1}(1.03) \) (answer in terms of \( \pi \)).

\[ f(x) = \tan^{-1}(x) \]

\[ f'(x) = \frac{1}{1 + x^2} \]

Approx near the known value \( x = 1 \):

\[ f(1) = \frac{\pi}{4} \]

\[ f'(1) = \frac{1}{2} \]

\[ \frac{\pi}{4} + \frac{1}{2} \cdot 0.03 \]

b) Find the tangent line to \( y^2 = x^2 - x \) at the point \( (2, \sqrt{6}) \).

\[ Z_y \cdot \frac{dy}{dx} = 3x^2 - 1 \]

\[ \frac{dy}{dx} = \frac{3x^2 - 1}{2y} \]

At \( (2, \sqrt{6}) \),

\[ \frac{dy}{dx} = \frac{3 \cdot 2^2 - 1}{2\sqrt{6}} = \frac{11}{2\sqrt{6}} \]

\[ (y - \sqrt{6}) = \frac{11}{2\sqrt{6}} \cdot (x - 2) \]
12. (graphing) Sketch the graph of \( f(x) = x \cdot e^{-x^2} \). You may assume that \( \lim_{x \to \infty} f(x) = 0 \) (we haven't covered how to evaluate these limits in detail).

- Determine local extrema, and where it's increasing/decreasing.
- Determine where it's concave up/down.

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) &= e^{-x^2} + x(-2x)e^{-x^2} \\
&= (1-2x^2)e^{-x^2}
\end{align*}
\]

Increasing: \( 1 = 2x^2 \) \( \Rightarrow \) \( x = \pm \sqrt{\frac{1}{2}} \)

Decreasing: \( 1 < 2x^2 \) \( \Rightarrow \) \( -1/\sqrt{2} < x < 1/\sqrt{2} \)

Concavity: \( f''(x) \) then the same sign as \(-2x(3-2x^2)\)

\[
\begin{align*}
f''(x) &= -4x(3-2x^2) = -4x(\sqrt{2}+x)(\sqrt{2}-x),
\end{align*}
\]

Rough sketch:

13. (graphing) Sketch the graph of \( f(x) = \frac{-3x^2+4x}{x^2-4} \).

- Find & classify vertical asymptotes (find limits on both sides).
- Find horizontal asymptotes.

Discontinuous at \( x^2-4 = 0 \), i.e., \( x = \pm 2 \).

\[
\begin{align*}
\lim_{x \to -2^-} f(x) &= \frac{-20}{0^+} = -\infty \\
\lim_{x \to -2^+} f(x) &= \frac{-20}{0^-} = \infty
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 2^-} f(x) &= \frac{-4}{0^-} = \infty \\
\lim_{x \to 2^+} f(x) &= \frac{-4}{0^+} = -\infty.
\end{align*}
\]

Vertical asymptotes at \( x = \pm 2 \).

Horizontal:

\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \infty} \frac{-3+4/x}{1-4/x^2} = \frac{-3+0}{1-0} = -3
\]

Horizontal asymptote at \( y = -3 \)....

rough sketch w/ this information:
A certain town has a workforce of 50,000 people, of whom 10% are unemployed. Suppose that the workforce is growing at a rate of 900 people per month, and 400 new jobs are being created per month. What is the rate of change of the unemployment rate (expressed in percent per month)?

Define:

\[ W(t) = \text{# people in the workforce} \]
\[ U(t) = \text{portion unemployed (between 0 and 1)} \]
\[ J(t) = \text{# people employed} \]

These are related by

\[ W(t) - W(t) \cdot U(t) = J(t) \]

Differentiate:

\[ W' - W' U - W \cdot U' = J' \]
\[ W' = W' U - J' = W \cdot U' \]
\[ \frac{W' - W' U - J'}{W} = U' \]

We are given

\[ W(0) = 50,000 \quad \text{and} \quad W'(0) = 900 \]
\[ U(0) = 0.1 \quad \Rightarrow \quad J(0) = 45,000 \]
\[ J'(0) = 400 \]

So at \( t = 0 \):

\[ U'(0) = \frac{900 - 900 \cdot 0.1 - 400}{50,000} \]
\[ = \frac{900 - 90 - 400}{50,000} \]
\[ = \frac{410}{50,000} \]
\[ = \frac{82}{10,000} = 0.0082 \]

so unemployment is increasing at a rate of 0.82% per month.