1. Use a quadratic approximation (order 2 Taylor approximation) of the function $\sqrt{x}$ (around a suitably chosen center) to find a rational number approximating $\sqrt{101}$. Use a calculator to compute the exact value, and determine how large the error is in your approximation.

2. Evaluate each of the following sums. You do not have to show that the sum converges.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n \cdot n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 4^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot 4^n}$

3. Evaluate each of the following sums (answer as a function of x). You do not have to show that the sum converges.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n$ (assume $x \geq 0$)

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^{3n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n)!} x^n$ (assume $x \geq 0$)

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^7}{n!} (x-7)^n$

4. Find the Taylor series of each function, around the specified center.

   (a) $\dfrac{1}{1-x}$ around $x = 0$.

   (b) $x^2 e^x$ around $x = 0$.

   (c) $\sin x$ around $x = \frac{\pi}{2}$.

   (d) $\displaystyle\int_0^x e^{-t^2}\, dt$ around $x = 0$.

5. Find the sixth order Taylor approximation of $e^{2x} \tan^{-1} x$ around $x = 0$. (*Hint*: avoid taking derivatives by multiplying two Taylor series together)

6. Give a series which sums to $\displaystyle\int_0^1 e^{-x^2/2}\, dx$.

7. Determine the radius of convergence of each of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!} x^{2n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n} x^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{8n+14} x^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

8. Suppose that $f(x)$ is a function with Taylor series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ around $x = 0$ (the numbers a_n are constants). Find a function whose Taylor series around $x = 0$ is $\displaystyle\sum_{n=0}^{\infty} n \cdot a_n x^n$ (your answer will be in terms of the function $f(x)$).

9. A certain circuit, which includes a resistor, a capacitor, and an inductor, is attached to alternating current. The charge $Q(t)$ on the capacitor after t seconds is described by the following initial value problem.

$$\begin{aligned} Q''(t) + 2Q'(t) + Q(t) &= \cos t \\ Q(0) &= 0 \\ Q'(0) &= 0 \end{aligned}$$

This differential equation is not of any of the types we have learned how to solve in this class (although you would learn how to write an exact solution in a course on differential equations). The goal of this problem is instead to obtain an approximation for $Q(t)$, valid for small values of t.

(a) Determine $Q''(0)$, by setting $t = 0$ in the differential equation.

(b) Differentiate both sides of the differential equation, to obtain

$$Q'''(t) + 2Q''(t) + Q'(t) = -\sin t.$$

Set $t = 0$ in this new equation, and determine the value of $Q'''(0)$ (you will need the result of part (a)).

(c) Using the same technique, determine the values of $Q^{(4)}(0)$ and $Q^{(5)}(0)$.

(d) Use the answers you found above, find a fifth order Taylor approximation for $Q(t)$.

(e) Give an approximation of $Q(1)$, the charge on the capacitor after 1 second.

Note. Although this particular differential equation can be solved exactly (using techniques we won't discuss in this course), many differential equations that arise in practice cannot. However, the technique outlined above can often be used to obtain good approximations over short time periods (in this case, near $t = 0$). Next week, we will use Fourier series to study differential equations like this one in the long term (after the special behavior near $t = 0$ has disappeared).