P. Set 8 Solutions

1) a) Observe that

\[ 0 \leq \frac{\sqrt{n}}{n^2+4} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}, \]

and \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) converges (e.g. by the integral test; this is a "p-series" with \( p > 1 \)). Therefore by comparison,

\[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+4} \] converges as well.

b) Observe that

\[ 0 \leq \frac{e^{x}}{x-1} \leq e^{-x} \quad \text{for } x \geq 2, \quad \text{(since } x-1 \geq 1) \]

and \( \int_{2}^{\infty} e^{-x} \, dx \) converges, since

\[ \int_{2}^{\infty} e^{-x} \, dx = [-e^{-x}]_{2}^{\infty} = -e^{-\infty} + e^{-2} = e^{-2}. \]

Therefore by comparison, \( \int_{2}^{\infty} \frac{e^{-x}}{x-1} \, dx \) converges as well.

c) Notice that for \( n \geq 1 \), \( 0 \leq \frac{1}{n} \leq \frac{\pi}{2} \), so \( 0 \leq \cos\left(\frac{1}{n}\right) \leq 1 \). Therefore \( 0 \leq \frac{1}{n^2} \cos\left(\frac{1}{n}\right) \leq \frac{1}{n^2} \). Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges (p-series w/ \( p = 2 > 1 \)), \( \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{1}{n}\right) \) converges as well. 

by comparison.
d) Observe that for $x > 2$,

$$0 \leq x^{-x} \leq x^{-2}$$

and $\int_{2}^{\infty} x^{-2} \, dx$ converges, so by comparison $\int_{2}^{\infty} x^{-x} \, dx$ (and therefore also $\int_{2}^{\infty} x^{-x} \, dx$) converges.

\[ \begin{align*}
\text{2a) } \quad & \lim_{n \to \infty} \frac{11^n n!}{n^n n!} = \lim_{n \to \infty} \frac{11^n}{n^n} \cdot \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{11^n}{n+1} = \frac{11^n}{\infty} = 0.
\text{This is less than 1, so } \sum_{n=0}^{\infty} \frac{11^n}{n!} \text{ converges.}
\end{align*} \]

\[ \begin{align*}
\text{2b) } \quad & \lim_{m \to \infty} \left| \frac{(-1)^m m!}{1000^m} \cdot \frac{m!}{1000^m} \right| = \lim_{m \to \infty} \frac{(m+1)!}{m!} \cdot \frac{1000^m}{1000^m} = \lim_{m \to \infty} \frac{m+1}{1000} = \frac{\infty}{1000} = \infty. \text{ (diverges to infinity).}
\text{Therefore the series } \sum (-1)^m \cdot \frac{m!}{1000^m} \text{ diverges.}
\end{align*} \]

\[ \begin{align*}
\text{2c) } \quad & \lim_{n \to \infty} \left| \frac{5^{n+1}}{(n+1)100} \cdot \frac{5^n}{n100} \right| = \lim_{n \to \infty} \frac{5^{n+1}}{5^n} \cdot \frac{n^{100}}{(n+1)^{100}} = \lim_{n \to \infty} (5 \cdot \frac{n}{n+1})^{100}
= 5.
\text{This is greater than 1, so the series } \sum_{n=5}^{\infty} \frac{5^n}{n100} \text{ diverges.}
\end{align*} \]
\[
\lim_{k \to \infty} \left| \frac{(-1)^k \cdot \frac{(k+1)!}{(k+1)^{k+1}}}{(-1)^k \cdot \frac{k!}{k^k}} \right| = \lim_{k \to \infty} \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} \\
= \lim_{k \to \infty} \frac{k^k}{(k+1)^{k+1}} = \lim_{k \to \infty} \frac{k^k}{(k+1)^k} \\
= \lim_{k \to \infty} \left( \frac{k}{k+1} \right)^k \\

The limit of \( \frac{k}{k+1} \) is 1, and so this is an indeterminate form? 1^\infty.

One way to resolve it is to first take the logarithm:

\[
\lim_{k \to \infty} \left[ \ln \left( \left( \frac{k}{k+1} \right)^k \right) \right] = \lim_{k \to \infty} \left[ k \cdot (\ln k - \ln(k+1)) \right] \\
= \lim_{k \to \infty} \frac{\ln k - \ln(k+1)}{1/k}
\]

This is now of the form 0/0 (since \( \lim_{k \to \infty} (\ln k - \ln(k+1)) = 0 \)), so apply l'Hospital:

\[
\text{(previous limit)} = \lim_{k \to \infty} \frac{\frac{1}{k} - \frac{1}{(k+1)}}{-1/k^2} = \lim_{k \to \infty} \frac{1/(k(k+1))}{-1/k^2} \\
= \lim_{k \to \infty} \left( -\frac{k}{k+1} \right) = -1.
\]

Therefore the original limit is

\[
\lim_{k \to \infty} \left( \frac{k}{k+1} \right)^k = e^{-1} = 1/e.
\]

This is less than 1, so \( \sum_{k=1}^{\infty} (-1)^k \cdot \frac{k!}{k^k} \) converges.
a) Taking absolute values gives $\sum_{n=0}^{\infty} e^{-n}$, a geometric series with ratio $e^{-1} < 1$, which converges (to $\frac{1}{1-e^{-1}}$).
Hence $\sum_{n=0}^{\infty} (-1)^n e^{-n}$ is absolutely convergent.

b) The numbers $\frac{1}{n}$ are decreasing and tend to 0. This is an alternating series, so by the all. series test it converges.

The series of absolute values $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the integral test ($\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = [2\sqrt{x}]_1^\infty = \infty - 2 = \infty$).
So $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ is conditionally convergent.

c) The terms of this series do not approach 0, since
$$\lim_{n \to \infty} |(-1)^n \frac{n^{1/2}}{2n+1}| = \lim_{n \to \infty} \frac{n^{1/2}}{2n+1} = \frac{1}{2}$$
so this series diverges.

d) The absolute values $\frac{1}{\sqrt{k^2-2}}$ are decreasing and tend to 0, so by the alternating series test this series converges.
However, the sum of the absolute values does not converge:

\[ 0 \leq \frac{1}{k} \leq \frac{1}{\sqrt{k^2+1}} \quad \text{(since } \sqrt{k^2+1} < \sqrt{k^2+k}) \]

and \( \sum \frac{1}{k} \) diverges, so by the comparison test

\( \sum \frac{1}{\sqrt{k^2+1}} \) diverges as well.

Therefore \( \sum_{k=2}^{\infty} (-1)^k \frac{1}{\sqrt{k^2+1}} \) converges conditionally.

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a) Using the ratio test,

\[ \lim_{n \to \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0 < 1 \]

so \( \sum_{n=0}^{\infty} \frac{1}{n!} \) converges. (In fact, it converges to the number e)

Other methods are possible; for example, you could use the comparison test, with

\[ 0 \leq \frac{1}{n!} \leq \frac{1}{n(n-1)} \leq \frac{1}{(n-1)^2} \quad \text{for } n \geq 2 \]

and note that \( \sum \frac{1}{(n-1)!} \) converges.

b) Using the comparison test:

\[ \frac{n}{\sqrt{n+10^{2x}}} \geq \frac{n}{\sqrt{n+10^{2x+1} \cdot \sqrt{n}}} = \frac{n}{\sqrt{n}} \cdot \left(\frac{1}{1+10^{2x}}\right) \]

\[ = \left(\frac{1}{1+10^{2x}}\right) \cdot \sqrt{n} \]

and \( \sum \left(\frac{1}{1+10^{2x}}\right) \sqrt{n} \) diverges (e.g., by the integral test).
so by the comparison test $\sum \frac{n}{\sqrt{n} + 10^{10}}$ diverges as well.

Another method:

$$\lim_{n \to \infty} \frac{n}{\sqrt{n} + 10^{10}} = \lim_{n \to \infty} \frac{\sqrt{n}}{1 + 10^{10} / n} = \frac{\infty}{1 + 0} = \infty.$$ 

So since the terms don't converge to 0, the series diverges (the "$n$" term test).

c) This is an alternating series. The magnitudes of the terms are decreasing ($\frac{1}{3n+4} < \frac{1}{3n+1}$) and tend to 0 ($\lim_{n \to \infty} \frac{1}{3n+1} = 0$), so the series converges by the alternating series test.

d) Using the integral test:

$$\int_{1}^{\infty} \frac{1}{5x^3} \, dx = \left[ \frac{1}{5} \ln |5x^3| \right]_{1}^{\infty} = \frac{1}{5} \ln (\infty) - \frac{1}{5} \ln 8 = \infty$$

Hence $\sum_{n=1}^{\infty} \frac{1}{5n+3}$ diverges as well.

Another method: use the comparison test, with

$$\frac{1}{5x^3} \geq \frac{1}{5x+3x} = \frac{1}{8} \cdot \frac{1}{x}.$$ 

e) $\sum_{k=1}^{\infty} \frac{1}{x^3}$ converges since $\int_{1}^{\infty} \frac{1}{x^3} \, dx = \left[ \frac{-1}{2} \cdot \frac{1}{x^2} \right]_{1}^{\infty} = \frac{1}{2}$ converges (integral test).
f) Using the ratio test:

\[
limit_{k \to \infty} \left| \frac{(-1)^{k+1} \frac{10^{k+1}}{(2k+2)!}}{(-1)^k \frac{10^k}{(2k)!}} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{10^{k+1}}{10^k} \cdot \frac{(2k)!}{(2k+2)!} \right|
\]

\[
= \lim_{k \to \infty} \frac{10}{(2k+2)(2k+1)}
\]

\[
= \frac{10}{\infty \cdot \infty} = 0
\]

Since 0 < 1, this series converges.

\(5\) a) \[
\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} 0.999^{n-1} \cdot 0.001
\]

\[
= \frac{0.001}{1 - 0.999} = \frac{0.001}{0.001} = 1
\]

because this is a geometric series with
first term 0.001 and common ratio 0.999.

Physically, this means that the probability that the particle will decay on some day is 1 (i.e. it will decay eventually).
\[ \mu = \sum_{n=1}^{\infty} n \cdot p_n = \sum_{n=1}^{\infty} n \cdot 0.999^{n-1} \cdot 0.001 = 0.001 \cdot \sum_{n=1}^{\infty} n \cdot 0.999^{n-1} . \]

By Problem 3 of the last problem set,

\[ \sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{d}{dx} \sum_{n=1}^{\infty} x^n = \frac{d}{dx} \frac{x}{1-x} = \frac{1}{(1-x)^2} \]

and this series converges when \( x = 0.999 \) (e.g. by the ratio test, but you don’t need to show this on your homework), hence

\[ \sum_{n=1}^{\infty} n \cdot 0.999^{n-1} = \frac{1}{(1-0.999)^2} = \frac{1}{0.001^2} \]

and therefore

\[ \mu = 0.001 \cdot \frac{1}{0.001^2} = \frac{1}{0.001} \]

\[ \mu = 1000 \]

In words: if the chance a particle decays on a given day is \( \frac{1}{1000} \), then the expected number of days before it decays is 1000 (which should seem plausible).