P. Set 8 Solutions

1. a) Observe that

$$0 \leq \frac{\sqrt{n}}{n^2+4} \leqslant \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (e.g. by the integral test; this is a "p-series" with $p>1$). Therefore by comparison, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+4}$ converges as well.

b) Observe that

$$0 \leq \frac{e^{-x}}{x-1} \leq e^{-x} \qquad \text{for } x \geqslant 2, \text{ (since } x-1 \geqslant 1)$$

and $\int_2^{\infty} e^{-x}dx$ converges, since

$$\int_2^{\infty} e^{-x}dx = \left[-e^{-x}\right]_2^{\infty} = -e^{-\infty} + e^{-2} = e^{-2}.$$

Therefore by comparison, $\int_2^{\infty} \frac{e^{-x}}{x-1}dx$ converges as well.

c) Notice that for $n \geqslant 1$, $0 \leq \frac{1}{n} \leq 1 \leq \frac{\pi}{2}$, so $0 \leq \cos(\frac{1}{n}) \leq 1$. Therefore $0 \leq \frac{1}{n^2}\cos(\frac{1}{n}) \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series w/ $p=2>1$), $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(\frac{1}{n})$ converges as well, by comparison.

d) Observe that for $x \geq 2$,

$$0 \leq x^{-x} \leq x^{-2}$$

and $\int_2^\infty x^{-2}dx$ converges, so by comparison $\int_2^\infty x^{-x}dx$ (and therefore also $\int_1^\infty x^{-x}dx$) converges.

2 a)

$$\lim_{n\to\infty} \frac{11^{n+1}/(n+1)!}{11^n/n!} = \lim_{n\to\infty} \frac{11^{n+1}}{11^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{11}{n+1} = \frac{11}{\infty} = 0.$$

This is less than 1, so $\sum_{n=0}^{\infty} \frac{11^n}{n!}$ converges.

b)

$$\lim_{m\to\infty} \left| \frac{(-1)^{m+1} \frac{(m+1)!}{1000^{m+1}}}{(-1)^m \cdot \frac{m!}{1000^m}} \right| = \lim_{m\to\infty} \frac{(m+1)!}{m!} \cdot \frac{1000^m}{1000^{m+1}}$$

$$= \lim_{m\to\infty} \frac{m+1}{1000} = \frac{\infty}{1000} = \infty. \text{ (diverges to infinity).}$$

Therefore the series $\sum (-1)^m \cdot \frac{m!}{1000^m}$ diverges.

c)

$$\lim_{n\to\infty} \left| \frac{\frac{5^{n+1}}{(n+1)^{100}}}{\frac{5^n}{n^{100}}} \right| = \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \cdot \frac{n^{100}}{(n+1)^{100}} = \lim_{n\to\infty} \left(5 \cdot \left(\frac{n}{n+1}\right)^{100}\right)$$

$= 5.$

This is greater than 1, so the series $\sum_{n=5}^{\infty} \frac{5^n}{n^{100}}$ diverges.

d) $$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}\cdot\frac{(k+1)!}{(k+1)^{k+1}}}{(-1)^k\cdot\frac{k!}{k^k}}\right| = \lim_{k\to\infty}\frac{(k+1)!}{k!}\cdot\frac{k^k}{(k+1)^{k+1}}$$

$$= \lim_{k\to\infty}(k+1)\cdot\frac{k^k}{(k+1)^{k+1}} = \lim_{k\to\infty}\frac{k^k}{(k+1)^k}$$

$$= \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k$$

The limit of $\frac{k}{k+1}$ is 1, and so this is an indeterminate form 1^∞.
One way to resolve it is to first take the logarithm:

$$\lim_{k\to\infty}\left[\ln\left(\left(\frac{k}{k+1}\right)^k\right)\right] = \lim_{k\to\infty}\left[k\cdot(\ln k-\ln(k+1))\right]$$

$$= \lim_{k\to\infty}\frac{\ln k-\ln(k+1)}{1/k}$$

this is now of the form $0/0$ (since $\lim_{k\to\infty}(\ln k-\ln(k+1))$ $= \lim_{k\to\infty}\ln\frac{k}{k+1} = \cancel{\lim}\ \ln 1 = 0$), so apply l'hôpital:

$$(\text{previous limit}) = \lim_{k\to\infty}\frac{1/k - 1/(k+1)}{-1/k^2} = \lim_{k\to\infty}\frac{1/(k(k+1))}{-1/k^2}$$

$$= \lim_{k\to\infty}\left(-\frac{k}{k+1}\right) = -1.$$

Therefore the original limit is

$$\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k = e^{-1} = 1/e.$$

This is less than 1, so $\sum_{k=1}^{\infty}(-1)^k\cdot\frac{k!}{k^k}$ <u>converges</u>.

3

a) Taking absolute values gives $\sum_{n=0}^{\infty} e^{-n}$, a geometric series w/ ratio $e^{-1}<1$, which converges (to $\frac{1}{1-e^{-1}}$).

Hence $\sum_{n=0}^{\infty}(-1)^n e^{-n}$ is absolutely convergent.

b) The numbers $\frac{1}{\sqrt{n}}$ are decreasing and tend to 0. This is an alternating series, so by the alt. series test it converges.

The series of absolute values $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges by the integral test ($\int_1^{\infty}\frac{1}{\sqrt{x}}dx = [2\sqrt{x}]_1^{\infty} = \infty - 2 = \infty$).

So $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{\sqrt{n}}$ is conditionally convergent.

c) The terms of this series do not approach 0, since

$$\lim_{l\to\infty}\left|(-1)^l\cdot\frac{l+1}{2l+1}\right| = \lim_{l\to\infty}\frac{l+1}{2l+1} = \frac{1}{2}$$

so this series diverges.

d) The absolute values $\frac{1}{\sqrt{k^2-2}}$ are decreasing & tend to 0, so by the alternating series test this series converges.

However, the sum of the absolute values does not converge:

$$0 \le \frac{1}{k} \le \frac{1}{\sqrt{k^2-\cancel{4}2}} \qquad \left(\text{since } \sqrt{k^2-4} < \sqrt{k^2} = k\right)$$

and $\sum \frac{1}{k}$ diverges, so by the comparison test

$\sum \frac{1}{\sqrt{k^2-\cancel{4}2}}$ diverges as well.

Therefore $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{1}{\sqrt{k^2-2}}$ <u>converges conditionally</u>.

(4)

a) Using the <u>ratio test</u>,

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

so $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ <u>converges</u>. (in fact, it converges to the number e)

Other methods are possible; for example you could use the comparison test, with

$$0 \le \frac{1}{n!} \le \frac{1}{n(n-1)} \le \frac{1}{(n-1)^2} \quad \text{for } n \ge 2$$

and note that $\sum \frac{1}{(n-1)^2}$ converges.

b) Using the comparison test:

$$\frac{n}{\sqrt{n}+10^{27}} \ge \frac{n}{\sqrt{n}+10^{27}\sqrt{n}} = \frac{n}{\sqrt{n}} \cdot \left(\frac{1}{1+10^{27}}\right)$$
$$= \left(\frac{1}{1+10^{27}}\right) \cdot \sqrt{n}$$

and $\sum \left(\frac{1}{1+10^{27}}\right)\sqrt{n}$ diverges (e.g. by the integral test).

so by the comparison test $\sum \frac{n}{\sqrt{n}+10^{27}}$ <u>diverges</u> as well.

Another method:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n}+10^{27}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+10^{27}/\sqrt{n}} = \frac{\infty}{1+0} = \infty.$$

so since the terms don't converge to 0, the series diverge (the "n^{th} term test").

c) This is an alternating series. The magnitudes of the terms are decreasing $\left(\frac{1}{3n+4} < \frac{1}{3n+1}\right)$ and tend to 0 $\left(\lim_{n\to\infty} \frac{1}{3n+1} = 0\right)$, so the series <u>converges</u> by the alternating series test.

d) Using the integral test:

$$\int_1^\infty \frac{1}{5x+3}\,dx = \left[\frac{1}{5}\ln|5x+3|\right]_1^\infty$$

$$= \frac{1}{5}\ln(\infty) - \frac{1}{5}\ln 8 = \infty$$

hence $\sum_{m=1}^{\infty} \frac{1}{5m+3}$ <u>diverges</u> as well.

Another method: use the comparison test, with

$$\frac{1}{5x+3} \geqslant \frac{1}{5x+3x} = \frac{1}{8}\cdot\frac{1}{x}.$$

e) $\sum_{l=1}^{\infty} \frac{1}{l^3}$ <u>converges</u> since $\int_1^\infty \frac{1}{x^3}dx = \left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_1^\infty = \frac{1}{2}$ converges (integral test).

f) Using the ratio test:

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}\cdot\frac{10^{k+1}}{(2k+2)!}}{(-1)^k\cdot\frac{10^k}{(2k)!}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{(-1)^k}\cdot\frac{10^{k+1}}{10^k}\cdot\frac{(2k)!}{(2k+2)!}\right|$$

$$=\lim_{k\to\infty} 10\cdot\frac{1}{(2k+2)(2k+1)}$$

$$=\frac{10}{\infty\cdot\infty}=0$$

Since $0<1$, this series <u>converges.</u>

5 a)

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} 0.999^{n-1}\cdot 0.001$$

$$=\frac{0.001}{1-0.999}=\frac{0.001}{0.001}=1$$

because this is a geometric series with first term 0.001 and common ratio 0.999.

Physically, this means that the probability that the particle will decay on <u>some</u> day is 1 (i.e. it will decay eventually).

b)

$$\mu = \sum_{n=1}^{\infty} n \cdot p_n = \sum_{n=1}^{\infty} n \cdot 0.999^{n-1} \cdot 0.001$$

$$= 0.001 \cdot \sum_{n=1}^{\infty} n \cdot 0.999^{n-1}.$$

By problem 3 of the last problem set,

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{d}{dx} \sum_{n=1}^{\infty} x^n$$

$$= \frac{d}{dx} \frac{x}{1-x} = \frac{1}{(1-x)^2}$$

and this series converges when $x = 0.999$ (e.g. by the ratio test, but you don't need to show this on your homework), hence

$$\sum_{n=1}^{\infty} n \cdot 0.999^{n-1} = \frac{1}{(1-0.999)^2} = \frac{1}{0.001^2}$$

and therefore

$$\mu = 0.001 \cdot \frac{1}{0.001^2} = \frac{1}{0.001}$$

$$\boxed{\mu = 1000}$$

In words: if the chance a particle decays on a given day is $1/1000$, then the expected number of days before it decays is 1000 (which should seem plausible).