Because this problem set was posted late, it will not be due until Monday. However, it will not require any material from Friday's class.

1. Evaluate each of the following series.

   (a) $6 + 2 + \frac{2}{3} + \frac{2}{9} + \cdots$

   (b) $\sum_{\ell=0}^{\infty} \frac{9}{10^\ell}$

   (c) $\sum_{\ell=1}^{\infty} \frac{9}{10^\ell}$

   (d) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{9^k}$

2. Write each sum in closed form. Your answer will be in terms of x. You may assume that the sum converges (it will converge for some values of x but not for others).

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{x^n}$

   (b) $\sum_{\ell=0}^{\infty} (-1)^\ell x^{\ell+1}$

   (c) $\sum_{k=1}^{\infty} x^{3k}$

   (d) $\sum_{k=1}^{\infty} (-1)^{k-1} \cdot x^{3k}$

3. Express the series $\sum_{n=1}^{\infty} n \cdot x^{n-1}$ in closed form (for those x that make the series converge).

4. Express the series $\sum_{n=1}^{\infty} n \cdot x^{2n}$ in closed form (for those x that make the series converge).

5. Evaluate $\sum_{n=1}^{\infty} n \cdot (2/3)^n$. You may assume that the series converges.

6. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{3}\right)^n$. You may assume that the series converges.

   Hint. Replace $\frac{1}{3}$ with x, and write the resulting series as the integral of some other series.

7. Find an infinite series of rational numbers which sums to π. You do not need to show that your series converges.

8. Find an infinite series of rational numbers which sums to $\arctan(\frac{1}{3})$. You do not need to show that your series converges.

9. **SKIP THIS PROBLEM. It has been moved to the next problem set.** Determine whether or not the improper integral $\int_2^\infty \frac{e^{-x}}{x-1}\,dx$ converges. You do not need to evaluate this integral.

10. **SKIP THIS PROBLEM. It has been moved to the next problem set.** Determine whether or not the improper integral $\int_1^\infty x^{-x}\,dx$ converges. You do not need to evaluate this integral.

11. Use the integral test to determine whether the series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^7}$

(c) $\displaystyle\sum_{\ell=10}^{\infty} \frac{1}{\ell(\ln \ell)^2}$

(d) $\displaystyle\sum_{\ell=20}^{\infty} \frac{1}{\ell \ln \ell}$

12. Let n be a positive integer. Show that

$$\int_0^{\infty} x^n e^{-x}\, dx = n \int_0^{\infty} x^{n-1} e^{-x}\, dx.$$

Note. In fact, this integral is equal to $n!$ (n factorial) for any nonnegative integer n, as you can convince yourself by looking at the case $n = 1$ and using the result of this problem. If n is replaced with non-integers (e.g. $n = 1/2$), this gives a way to extend the factorial function to non-integers; this idea is very common in probability and statistics (you can learn more by looking up the "gamma function").

13. (Charging a capacitor with a battery) When a capacitor, an inductor, and a resistor are arranged in a circuit (in sequence) with a battery, the charge $Q(t)$ on the capacitor can be described by the following initial value problem.

$$\begin{aligned} V &= LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) \\ Q(0) &= 0 \ \ \text{(no charge initially)} \\ Q'(0) &= 0 \ \ \text{(no current initially)} \end{aligned}$$

Here, V is the voltage of the battery (in Volts), L is the inductance (measured in Henries), R is the resistance (in Ohms), and C is the capacitance (in Farads). Assume that $V = 1, L = 1$, and $C = 0.01$.

(a) Solve this initial value problem for $Q(t)$ in the case $R = 0$ (no resistance in the circuit).

(b) Solve this initial value problem in the case $R = 16$.

(c) Solve this initial value problem in the case $R = 25$.

(d) Notice that when the resistance is small, the solution that you found involves oscillation (there are sines and cosines), while when the resistance is large it does not. Find the smallest amount of resistance such that there is no oscillation in the solution.