P. Set 6 Solutions

(1)

$f' = 4f^2 \cdot t$

$\frac{1}{f^2} \cdot f' = 4t$

$\int \frac{1}{f^2} df = \int 4t\,dt$

$-1/f = 2t^2 + C$

$\Rightarrow \quad f(t) = -\frac{1}{2t^2+C}$

(general sol'n of diff Eq).

initial data:

$1 = f(0) = -\frac{1}{-2\cdot 0 + C}$

$\Rightarrow \; -1 = C$

so $\boxed{f(t) = \frac{1}{1-2t^2}}$

(2)

$\frac{dy}{dx} = 6 \cdot \frac{x^3}{y-1}$

$\int (y-1)\,dy = \int 6x^3 dx$

$\frac{1}{2}y^2 - y = \frac{3}{2}x^4 + C$

$y^2 - 2y = 3x^4 + 2C$

$y^2 - 2y - (3x^4 + 2C) = 0$

quad. formula:

$y = \frac{1}{2} \cdot \left[2 \pm \sqrt{4 + 4(3x^4+2C)}\right]$

$\underline{y = 1 \pm \sqrt{3x^4 + 2C + 1}}$

initial data:

$3 = y(0) = 1 \pm \sqrt{3 \cdot 0 + 2C + 1}$

$2 = \pm\sqrt{2C+1}$

so "$\pm$" is "$+$" and $2C+1 = 4$ for this IVP.

$\boxed{y = 1 + \sqrt{3x^4 + 4}}$

3

$$\frac{dy}{dt} = e^{-2y}\sqrt{1-t^2}$$

Trig sub: $t = \sin\theta$, $dt = \cos\theta\, d\theta$, $\sqrt{1-t^2} = \cos\theta$

$$\int e^{2y}\,dy = \int \sqrt{1-t^2}\,dt$$

$$\frac{1}{2}e^{2y} = \int \cos^2\theta\, d\theta = \int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$\frac{1}{2}e^{2y} = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$$\frac{1}{2}e^{2y} = \frac{1}{2}\sin^{-1}t + \frac{1}{4}\sin(2\cdot\sin^{-1}t) + C$$

$$e^{2y} = \sin^{-1}t + \frac{1}{2}\sin(2\cdot\sin^{-1}t) + 2C$$

$$2y = \ln\left(\sin^{-1}t + \frac{1}{2}\sin(2\cdot\sin^{-1}t) + 2C\right)$$

$$y = \frac{1}{2}\ln\left(\sin^{-1}t + \frac{1}{2}\sin(2\cdot\sin^{-1}t) + 2C\right)$$ gen'l sol'n to diff Eq.

initial data:

$$1 = \frac{1}{2}\ln(0+0+2C) = \frac{1}{2}\ln(2C)$$

$$\Rightarrow 2C = e^2.$$

so $\boxed{f(t) = \frac{1}{2}\ln\left(\sin^{-1}t + \frac{1}{2}\sin(2\cdot\sin^{-1}t) + e^2\right)}$

Note Using the fact $\sin(2\theta) = 2\sin\theta\cos\theta$, the term $\frac{1}{2}\sin(2\sin^{-1}t)$ could also be rewritten as $t\cdot\sqrt{1-t^2}$, if desired.

(4)

$$f'(x) = \frac{1+f(x)^2}{\sqrt{4-x^2}}$$

$$\frac{1}{1+f(x)^2}\cdot f'(x) = \frac{1}{\sqrt{4-x^2}}$$

$$\int \frac{1}{1+f^2}\,df = \int \frac{1}{\sqrt{4-x^2}}\,dx$$

$$\tan^{-1}(f(x)) = \sin^{-1}\left(\tfrac{x}{2}\right) + C$$

$$f(x) = \tan\left(\sin^{-1}\left(\tfrac{x}{2}\right) + C\right) \quad \text{gen'l sol'n.}$$

initial data:

$$1 = f(0) = \tan(0 + C)$$

$$\Rightarrow \quad C = \tfrac{\pi}{4}.$$

so $\boxed{f(x) = \tan\left(\sin^{-1}\left(\tfrac{x}{2}\right) + \tfrac{\pi}{4}\right)}$

(5)

$$y'' + 5y' + 6y = 6$$

$$y'' + 5y' + 6(y-1) = 0$$

Let $u = y-1$. Then

$u'' + 5u' + 6u$ (homog. linear)

$$\begin{cases} u(0) = 0 \\ u'(0) = 1 \end{cases}$$

char. eqn: $\lambda^2 + 5\lambda + 6 = 0$ ie. $(\lambda+2)(\lambda+3) = 0$

So gen'l sol'n for u is

$$u(t) = C\cdot e^{-2t} + D\cdot e^{-3t}$$

using the initial conditions for u:

$$0 = u(0) = C + D \qquad D = -C \qquad \text{so}$$
$$1 = u'(0) = -2C - 3D \qquad 1 = -2C + 3C = C \qquad C = 1, \; D = -1$$

hence $\quad u(t) = e^{-2t} - e^{-3t}$

and $\quad \boxed{y(t) = 1 + e^{-2t} - e^{-3t}}$

(6)

$$f'' + 4f' + 13f = 104$$
$$f'' + 4f' + 13(f-8) = 0$$

Let $u = f-8$, so that

$$u'' + 4u' + 13u = 0 \quad \text{(homog.)}$$
$$\begin{cases} u(0) = 3 \\ u'(0) = 0 \end{cases}$$

char. eqn. $\quad \lambda^2 + 4\lambda + 13 = 0$

sol'ns $\quad \lambda = -2 \pm 3i$

$\Rightarrow$ one complex sol'n is $\quad u(t) = e^{-2t}\cos(3t) + i\, e^{-2t}\sin(3t)$

$\Rightarrow$ two real sol'ns are $\quad e^{-2t}\cos(3t), \quad e^{-2t}\sin(3t)$

$\Rightarrow$ gen'l sol'n for u is $\quad u(t) = C\cdot e^{-2t}\cos(3t) + D\cdot e^{-2t}\sin(3t)$

$$u(0) = C + 0$$
$$u'(t) = -2Ce^{-2t}\cos(3t) - 3Ce^{-2t}\sin(3t) - 2De^{-2t}\sin(3t) + 3De^{-2t}\cos(3t)$$
$$u'(0) = -2C - 3C\cdot 0 - 2D\cdot 0 + 3D$$
$$= -2C + 3D$$

using initial conds. for u:

$$3 = u(0) = C \qquad\qquad C = 3$$
$$0 = u'(0) = -2C + 3D \qquad D = \tfrac{2}{3}C = 2$$

so $u(t) = 3e^{-2t}\cos(3t) + 2e^{-2t}\sin(3t)$

and $\boxed{f(t) = 8 + 3e^{-2t}\cos(3t) + 2e^{-2t}\sin(3t)}$

(7) The equation $f''(t) = -f(t)$ has general sol'n

$$f(t) = C\cdot\cos t + D\cdot\sin t$$

Hence, treating each coordinate as a function, the equation

$$\vec{a}(t) = -\vec{r}(t) \qquad \left(\text{where } \vec{a}(t) = \frac{d^2}{dt^2}\vec{r}(t)\right)$$

has general solution

$$\vec{r}(t) = \vec{C}\cdot\cos t + \vec{D}\cdot\sin t$$

where $\vec{C}, \vec{D}$ are two (constant) vectors. In terms of $\vec{C}$ and $\vec{D}$,

$$\vec{r}(0) = \vec{C}$$
$$\vec{v}(t) = -\vec{C}\cdot\sin t + \vec{D}\cdot\cos t$$
$$\vec{v}(0) = \vec{D}$$

hence ~~$\vec{C} = (1,0$~~ using the initial conditions,

$$(1,0,0) = \vec{r}(0) = \vec{C}$$
$$(0,2,3) = \vec{v}(0) = \vec{D}$$

so $\vec{r}(t) = (1,0,0)\cdot\cos t + (0,2,3)\cdot\sin t$

$$\boxed{\vec{r}(t) = (\cos t, 2\sin t, 3\cdot\sin t)}$$

8

$$\int_0^\infty x \cdot e^{-x^2}\,dx$$

$$= \lim_{b\to\infty} \int_0^b x\cdot e^{-x^2}\,dx \qquad \begin{matrix} u=-x^2 \\ du=-2x \end{matrix}$$

$$= \lim_{b\to\infty} \int_0^{-b^2} \left(-\tfrac{1}{2}\right) e^u\,du$$

$$= \lim_{b\to\infty} \left[-\tfrac{1}{2}e^u\right]_0^{-b^2}$$

$$= \lim_{b\to\infty} \left[-\tfrac{1}{2}e^{-b^2} + \tfrac{1}{2}\right] = \boxed{\tfrac{1}{2}}$$

9 $\displaystyle\int_0^\infty e^{-x}\cos(2x)\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos(2x)\,dx$

indef. integral:

$$\int e^{-x}\cos(2x)\,dx \qquad \begin{matrix} u=\cos(2x) & dv=e^{-x}dx \\ du=-2\sin(2x) & v=-e^{-x} \end{matrix}$$

$$= -e^{-x}\cos(2x) - \int 2e^{-x}\sin(2x)\,dx \qquad \begin{matrix} u=\sin(2x) & dv=2e^{-x}dx \\ du=2\cos(2x) & v=-2e^{-x} \end{matrix}$$

$$= -e^{-x}\cos(2x) + 2e^{-x}\sin(2x) - 4\int e^{-x}\cos(2x)\,dx$$

$$\Rightarrow \quad 5\cdot\int e^{-x}\cos(2x)\,dx = 2e^{-x}\sin(2x) - e^{-x}\cos(2x)\ (+C)$$

$$\Rightarrow \quad \int e^{-x}\cos(2x)\,dx = \tfrac{2}{5}e^{-x}\sin(2x) - \tfrac{1}{5}e^{-x}\cos(2x)\quad(+C)$$

So $\displaystyle\int_0^b e^{-x}\cos(2x)\,dx = \left[\tfrac{2}{5}e^{-x}\sin(2x) - \tfrac{1}{5}e^{-x}\cos(2x)\right]_0^b$ *

$$= \tfrac{2}{5}e^{-b}\sin(2b) - \tfrac{1}{5}e^{-b}\cos(2b) - \tfrac{2}{5}\cdot e^0\cdot\sin 0 + \tfrac{1}{5}\cdot e^0\cdot\cos 0$$

$$= \tfrac{2}{5}e^{-b}\sin(2b) - \tfrac{1}{5}e^{-b}\cos(2b) + \tfrac{1}{5}$$

and

$$\int_0^\infty e^{-x}\cos(2x)\,dx$$

$$= \lim_{b\to\infty}\left[\tfrac{2}{5}e^{-b}\sin(2b) + -\tfrac{1}{5}e^{-b}\cos(2b) + \tfrac{1}{5}\right]$$

$$= \lim_{b\to\infty}\left(\frac{2\sin(2b)-\cos(2b)}{5e^{b}}\right) + \frac{1}{5}$$

since $2\sin(2b)-\cos(2b)$ is bounded and $\lim_{b\to\infty}(e^{b})=\infty$, this last limit equals 0

(more formally, one could apply the squeeze theorem here),

hence

$$\int_0^\infty e^{-x}\cos(2x)\,dx = \boxed{\frac{1}{5}}$$

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a) $$\sum_{n=1}^{5}(2n-1) = (2\cdot1-1)+(2\cdot2-1)+(2\cdot3-1)+(2\cdot4-1)+(2\cdot5-1)$$
$$= 1+3+5+7+9$$
$$= \boxed{25}$$

b) $$\sum_{l=0}^{3}(l+2)^2 = (0+2)^2+(1+2)^2+(2+2)^2+(3+2)^2$$
$$= 2^2+3^2+4^2+5^2$$
$$= 4+9+16+25$$
$$= \boxed{54}$$

c) $\sum_{k=0}^{5} (-1)^k \cdot 2^{-k}$

$$= (-1)^0 \cdot 2^{-0} + (-1)^1 \cdot 2^{-1} + (-1)^2 \cdot 2^{-2} + (-1)^3 \cdot 2^{-3} + (-1)^4 \cdot 2^{-4} + (-1)^5 \cdot 2^{-5}$$

$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$$

$$= \frac{32-16+8-4+2-1}{32} = \boxed{\frac{21}{32}}$$

d) $\sum_{n=-2}^{100} (5) = \underbrace{5+5+5+\cdots+5}_{\text{one 5 for each } n \text{ in } -2,\cdots,100}$

(103 such n)

$$= 5 \cdot 103$$

$$= \boxed{515}$$

(11) a) $1+2+\cdots+100 = \boxed{\sum_{n=1}^{100} n}$

b) $\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{20!} = \boxed{\sum_{k=0}^{20} \frac{1}{k!}}$

c) $1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{101} = \frac{1}{2\cdot 0+1} - \frac{1}{2\cdot 1+1} + \frac{1}{2\cdot 2+1} - \cdots + \frac{1}{2\cdot 50+1}$

$$= \frac{(-1)^0}{2\cdot 0+1} + \frac{(-1)^1}{2\cdot 1+1} + \frac{(-1)^2}{2\cdot 2+1} + \cdots + \frac{(-1)^{50}}{2\cdot 50+1}$$

$$= \boxed{\sum_{k=0}^{50} \frac{(-1)^k}{2k+1}}$$

d)

$$\frac{15}{2^0} + \frac{14}{2^1} + \frac{13}{2^2} + \cdots + \frac{0}{2^{15}}$$

$$= \frac{15-0}{2^0} + \frac{15-1}{2^1} + \frac{15-2}{2^2} + \cdots + \frac{15-15}{2^{15}}$$

$$= \boxed{\sum_{n=0}^{15} \left(\frac{15-n}{2^n}\right)}$$

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$n=1$: $\displaystyle\sum_{k=1}^{1} \frac{1}{k(k+1)} = \boxed{\frac{1}{2}}$

$n=2$: $\displaystyle\sum_{k=1}^{2} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \boxed{\frac{2}{3}}$

$n=3$: $\displaystyle\sum_{k=1}^{3} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{8}{12} + \frac{1}{12}$

$$= \frac{9}{12} = \boxed{\frac{3}{4}}$$

$n=4$: $\displaystyle\sum_{k=1}^{4} \frac{1}{k(k+1)} = \underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4}} + \frac{1}{4\cdot 5}$

$$= \frac{3}{4} + \frac{1}{20} = \frac{15}{20} + \frac{1}{20} = \frac{16}{20} = \boxed{\frac{4}{5}}$$

It is apparent from these examples that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$ appears to equal $\boxed{\frac{n}{n+1}}$. In fact, this is true for all n (one way to prove this ~~to~~ is to write $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and let the series "telescope."