PSet 5 Solutions

① a) $\lambda - 27 = 0$
b) $\lambda^3 + \lambda^2 + 1 = 0$
c) $\lambda^7 + \lambda = 0$
d) $\lambda^2 - 4\lambda + 16 = 0$

② a) Char. eqn. $\lambda^2 + 8\lambda + 7 = 0$
Solutions: $(\lambda+1)(\lambda+7) = 0 \Rightarrow \lambda = -1$ or $\lambda = -7$.
So a real solution is $\boxed{f(x) = e^{-x}}$ (another is e^{-7x})

b) Char. eqn. $\lambda^2 + 8\lambda + 16 = 0$
Solutions: $(\lambda+4)^2 = 0 \Rightarrow \lambda = -4$
So a real solution is $\boxed{f(x) = e^{-4x}}$ (repeated root ⇒ another is $x \cdot e^{-4x}$)

c) Char. eqn. $\lambda^2 + 8\lambda + 20 = 0$
Solutions: $\lambda = \frac{1}{2}(-8 \pm \sqrt{64-80}) = -4 \pm \sqrt{-4} = -4 \pm 2i$.
one Complex solution: $e^{(-4+2i)x} = e^{-4x} \cdot (\cos(2x) + i\sin(2x))$
so one real solution is $\boxed{e^{-4x}\cos(2x)}$ (another is $e^{-4x}\sin(2x)$).

d) Char. eqn. $\lambda^2 + 8\lambda + 116 = 0$
Solutions $\lambda = \frac{1}{2}(-8 \pm \sqrt{8^2 - 4 \cdot 116}) = -4 \pm \sqrt{16-116} = -4 \pm \sqrt{-100}$
$= -4 \pm 10i$
one complex sol'n is $e^{(-4+10i)x} = e^{-4x} \cdot (\cos(10x) + i \cdot \sin(10x))$
so one real solution is $\boxed{e^{-4x}\cos(10x)}$ (another is $e^{-4x}\sin(10x)$)

③ $f'''(x) + f(x)$ has char. eqn. $\lambda^3 + 1 = 0$, ie. $\lambda^3 = -1$.
Using polar form for -1, this is $\lambda^3 = e^{\pi i}$. One solution is $\lambda = e^{\pi i/3}$
(the others are $e^{-\pi i/3}$ & $e^{3\pi i/3} = e^{\pi i} = -1$).

In rectangular form,

$$e^{\pi i/3} = \cos(\tfrac{\pi}{3}) + i\cdot\sin(\tfrac{\pi}{3})$$
$$= \tfrac{1}{2} + i\cdot\tfrac{\sqrt{3}}{2}.$$

So a complex sol'n of the differential equation is

$$e^{(\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2})x} = e^{x/2}\cdot e^{i(\sqrt{3}/2)\cdot x}$$
$$= e^{x/2}\cdot\left(\cos(\tfrac{\sqrt{3}}{2}x) + i\sin(\tfrac{\sqrt{3}}{2}x)\right).$$

Taking the real part gives one real solution $\boxed{f(x) = e^{x/2}\cdot\cos(\tfrac{\sqrt{3}}{2}x).}$

Note. Another sol'n comes from the imag. part: $e^{x/2}\cdot\sin(\tfrac{\sqrt{3}}{2}x)$.
The value $\lambda = e^{-\pi i/3} = \tfrac{1}{2} - i\cdot\tfrac{\sqrt{3}}{2}$ gives the same two solutions (up to sign). The value $\lambda=-1$ gives $f(x)=e^{-x}$, though the problem asks for one of the other ones.
The general solution has three constants (third order eqn):

$$f(x) = C\cdot e^{-x} + D\cdot e^{x/2}\cos(\tfrac{\sqrt{3}}{2}x) + E\cdot e^{x/2}\cdot\sin(\tfrac{\sqrt{3}}{2}x).$$

(4) $f(x) = e^{-7x}\cos 2x$ could be obtained as the real part of

$$e^{-7x}\cos(2x) + i\cdot e^{-7x}\sin(2x)$$
$$= e^{(-7+2i)x}$$

so we're looking for a diff Eq whose char. eqn. has $\lambda=-7+2i$ as a solution. So the char. equation could be

$$(\lambda-(-7+2i))(\lambda-(-7-2i)) = 0$$
$$(\lambda+7-2i)(\lambda+7+2i) = 0$$
$$(\lambda+7)^2 - (2i)^2 = 0$$
$$\lambda^2 + 14\lambda + 49 + 4 = 0$$

or $\lambda^2+14\lambda+\cancel{45}53$

So the diff. Eq. could be $\boxed{f''(x)+14f'(x)+53f(x)=0}$

(5) The char. eqn. is $\lambda^2+d\lambda+k=0$,
which has solutions $\lambda=\frac{1}{2}\left(-d\pm\sqrt{d^2-4k}\right)$.
These are real precisely when the square root is not of a negative number; that is

$$\text{overdamped} \iff d^2-4k\geq 0$$
$$\iff d^2\geq 4k.$$

a) If $k=16$, then the spring is overdamped $\iff d^2\geq 4\cdot 16$,
ie. $\boxed{|d|\geq 8}$ ($d\geq 8$ is also a fine answer since d is positive for physical springs).

b) If $d=6$, then the spring is overdamped $\iff 6^2\geq 4k$,
ie. $\boxed{k\leq 9}$ (note: small values of k cause overdamping, while large values of d do).

(6) a) This is linear & homog. Char. eqn: $\lambda+5=0$, so one solution is e^{-5x}. Mult. by a constant, the general sol'n is $\boxed{f(x)=C\cdot e^{-5x}}$.

b) $f'(x) = 5\sin x$; take antiderivative.

$\Rightarrow f(x) = \int 5\sin x\,dx$

$\boxed{f(x) = -5\cos x + C}$

c) $f'(x) = 3f(x) \iff f'(x) - 3f(x) = 0.$

Linear & homog., char. $\lambda - 3 = 0$ ie. $\lambda = 3$.

So e^{3x} is one sol'n; gen'l sol'n is $\boxed{f(x) = C\cdot e^{3x}}$

d) $f'(x) = 3x^2$; take antiderivative

$f(x) = \int 3x^2\,dx \Rightarrow \boxed{f(x) = x^3 + C}$

7 a) We saw in (2) that e^{-x} & e^{-7x} are two sol'ns.

This is linear & homog., so gen'l sol'n is $\boxed{f(x) = C\cdot e^{-x} + D\cdot e^{-7x}}$

b) We saw e^{-4x}. Because $\lambda = -4$ was a double root, $x\cdot e^{-4x}$ is another sol'n. So the gen'l sol'n is $\boxed{f(x) = C\cdot e^{-4x} + D\cdot x\cdot e^{-4x}}$

c) We saw $e^{-4x}\cos(2x)$ & $e^{-4x}\sin(2x)$.

So gen'l sol'n is $\boxed{f(x) = C\cdot e^{-4x}\cos(2x) + D\cdot e^{-4x}\sin(2x)}$

d) We saw $e^{-4x}\cos(10x)$ & $e^{-4x}\sin(10x)$.

So gen'l sol'n is $\boxed{f(x) = C\cdot e^{-4x}\cos(10x) + D\cdot e^{-4x}\cdot\sin(10x)}$

(8) Use the gen'l sol'n from (7) in each part:

a) $0 = f(0) = C\cdot e^{-0} + D\cdot e^{-0}$

$\underline{0 = C + D}$

$6 = f'(0)$ and $f'(x) = -C\cdot e^{-x} - 7D\cdot e^{-7x}$

$\Rightarrow 6 = -C\cdot e^{-0} - 7D\cdot e^{-0}$

$= -C - 7D$

So solve	To obtain	Hence
$C + D = 0$	$C = 1$	$\boxed{f(x) = e^{-x} - e^{-7x}}$
$-C - 7D = 6$	$D = -1$	

b) $0 = f(0) = C\cdot e^{-0} + D\cdot 0\cdot e^{-0}$

$\Rightarrow \underline{0 = C}.$

$6 = f'(0)$ and $f'(x) = -4\cdot C\cdot e^{-4x} + D\cdot e^{-4x} + -4D\cdot x\cdot e^{-4x}$

$= (D - 4C)\cdot e^{-4x} - 4Dx\cdot e^{-4x}$

$\Rightarrow 6 = (D - 4C)\cdot 1 - 4\cdot D\cdot 0$

$\underline{6 = D - 4C}$

So solve	To obtain	Hence
$0 = C$	$C = 0$	$\boxed{f(x) = 6\cdot x e^{-4x}}$
$6 = D - 4C$	$D = 6$	

c) $0 = f(0) = C \cdot 1 \cdot 1 + D \cdot 1 \cdot 0$

$\Rightarrow \underline{0 = C}.$

$6 = f'(0)$ and $f'(x) = -4Ce^{-4x}\cos(2x) - 2Ce^{-4x}\sin(2x)$
$\quad -4De^{-4x}\sin(2x) + 2De^{-4x}\cos(2x)$
$\quad = (-4C+2D)e^{-4x}\cos(2x) + (-2C-4D)e^{-4x}\sin(2x)$

$\Rightarrow 6 = (-4C+2D) \cdot 1 \cdot 1 + (-2C-4D) \cdot 1 \cdot 0$

$\underline{6 = -4C+2D}$

Solve $0=C$, $6=-4C+2D$
to obtain $C=0$, $D=3$.

$$\boxed{f(x) = 3 \cdot e^{-4x}\sin(2x)}$$

d) $0 = f(0) = C \cdot 1 \cdot 1 + D \cdot 1 \cdot 0$

$\Rightarrow \underline{0 = C}$

$f'(x) = -4Ce^{-4x}\cos(10x) - 10Ce^{-4x}\sin(10x)$
$\quad -4De^{-4x}\sin(10x) + 10De^{-4x}\cos(10x)$
$\quad = (-4C+10D)e^{-4x}\cos(10x) + (-10C-4D)e^{-4x}\sin(10x)$

$6 = f'(0) = (-4C+10D) \cdot 1 \cdot 1 + (-10C-4D) \cdot 1 \cdot 0$
$\quad = -4C+10D$.

Solve $0=C$, $6=-4C+10D$ to obtain $C=0$, $D=3/5$.

$$\boxed{f(x) = \frac{3}{5}e^{-4x}\sin(10x)}$$