P. Set 4 Solutions

1. a) $\left( r, \Theta \right) = (-2, \pi/27) \quad \Rightarrow \quad \left( r, \Theta \right) = (2, -\frac{26}{27}\pi)$

b) $\left( r, \Theta \right) = (2, 3\pi) \quad \Rightarrow \quad \left( r, \Theta \right) = (2, \pi)$

c) $\left( r, \Theta \right) = (4, -\pi/9) \quad \Rightarrow \quad \left( r, \Theta \right) = (4, \frac{17}{9}\pi)$

d) $\left( r, \Theta \right) = (5, 8\pi/9) \quad \Rightarrow \quad \left( r, \Theta \right) = (5, -\frac{\pi}{9})$

(many other answers are possible for each part)

2. $r = 2\sqrt{\cos 2\Theta}, \quad -\frac{\pi}{3} \leq \Theta \leq \frac{\pi}{3}$

a) $r^2 = 4\cos(2\Theta) \quad \Rightarrow \quad r = \pm 2\cos\Theta$

$\Rightarrow r^4 = 4(r^2\cos^2\Theta - 4\sin^2\Theta)^2$ \quad \Rightarrow \quad $\left( x^2 + y^2 \right)^2 = 4x^2 - 4y^2$

Note. This equation gives more points than those from $-\frac{\pi}{3} \leq \Theta \leq \frac{\pi}{3}$; these points appear for values with $\frac{3\pi}{2} \leq \Theta \leq \frac{5\pi}{2}$. Other values of $\Theta$ would not give real values of $r$.

b) Area $= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\Theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} (2\cos\Theta)^2 d\Theta$

$= 2 \int_{-\pi/4}^{\pi/4} \cos(2\Theta) d\Theta = \left[ \sin(2\Theta) \right]_{-\pi/4}^{\pi/4} = 2$

c) $\frac{dr}{d\Theta} = 2 \cdot \frac{1}{2\sqrt{\cos 2\Theta}} \cdot \frac{d}{d\Theta} (\cos 2\Theta) = \frac{1}{\sqrt{\cos 2\Theta}} \cdot (-\sin 2\Theta) \cdot 2 = -2 \cdot \frac{\sin 2\Theta}{\sqrt{\cos 2\Theta}}$
\[ r^2 + \left( \frac{dr}{d\theta} \right)^2 = 4 \cos 2\theta + 4 \cdot \frac{\sin^2 2\theta}{\cos 2\theta} = 4 \left( \frac{\cos^2 2\theta}{\cos 2\theta} + \frac{\sin^2 2\theta}{\cos 2\theta} \right) \]

\[ = \frac{4}{\cos 2\theta} = 4 \sec 2\theta \]

\[ \Rightarrow \ \text{arc-length} = \int_{-\pi/4}^{\pi/4} \sqrt{4 \sec 2\theta} \, d\theta = 2 \int_{-\pi/4}^{\pi/4} \sqrt{\sec 2\theta} \, d\theta \]

\[ = \int_{-\pi/2}^{\pi/2} \sqrt{\sec u} \, du \quad (\text{where } u = 2\theta) \]

using a computer, this is \( \approx 5.244 \).

\[3\]

\[ (x-A)^2 + (y-B)^2 = A^2 + B^2 \]

\[ x^2 - 2xA + A^2 + y^2 - 2yB + B^2 = A^2 + B^2 \]

\[ x^2 + y^2 - 2xA - 2yB = 0 \]

\[ \Rightarrow \ \ r^2 - 2r\cos\phi \cdot A - 2r\sin\phi \cdot B = 0 \]

\[ \Leftrightarrow \ \ r = 0 \text{ or } \ r = 2A \cos \phi - 2B \sin \phi = 0 \]

\[ \square \ \ r = 2A \cdot \cos \phi + 2B \cdot \sin \phi \]

This problem originally had a misprint, and read:

\[ (x-A)^2 + (y-B)^2 = A^2 + B^2 \]

which is equivalent to

\[ 2x^2 - 2x(A+B) = 0 \]

\[ \Rightarrow \ \ x = 0 \text{ or } x = A+B, \ a \ \text{pair of vertical lines.} \]

The second line has polar equation \( r = (A+B)/\cos \phi \), while the first \( (x=0) \) has no equation of the form \( r = F(\phi) \).

Due to the misprint, the question will also accept \( r = (A+B)/\cos \phi \).
When \( n \) is odd there are \( n \) petals; when \( n \) is even there are \( \frac{2n}{n} \).

[The reason for this is that the petals end at the \( Zn \) polar point:

\((1, \frac{\pi}{n}), (-1, \frac{3\pi}{n}), (1, \frac{5\pi}{n}), (-1, \frac{7\pi}{n}), \ldots\),

\((1, \frac{4n-3\pi}{2n}), (-1, \frac{4n-\pi}{2n})\).]
Then points contain repeats:

\[(1, \frac{\pi}{2n}) \quad \text{is the same point as} \quad (-1, \frac{(2n+1)\pi}{2n})\]

\[(1, \frac{3\pi}{2n}) \quad \text{and} \quad (1, \frac{(2n+3)\pi}{2n})\]

(etc.)

so each lobe is covered *twice* as \(\theta\) ranges through \([0, 2\pi)\).

The area of one lobe is the area over \(0 \leq \theta \leq \frac{\pi}{n}\), which is

\[
\int_0^{\pi/n} \frac{1}{2} r^2 \, d\theta
= \frac{1}{2} \int_0^{\pi/n} \sin^2(n\theta) \, d\theta
= \frac{1}{2} \int_0^{\pi/n} \sin^2 u \, du
= \frac{1}{2n} \int_0^{\pi} \frac{1}{2} (1 - \cos 2u) \, du
= \frac{1}{2n} \left[ \frac{1}{2} u - \frac{1}{4} \sin 2u \right]_0^{\pi}
= \frac{1}{2n} \cdot \frac{\pi}{2} = \frac{\pi}{4n}
\]

Another method: \(\frac{1}{2} \int_0^{2\pi} \sin^2(n\theta) \, d\theta = \frac{\pi}{2} \), and there are \(2n\) lobes covered from 0 to \(2\pi\) (if \(n\) is odd there are \(n\) lobes, but each one is \(\pi/n\) swept out twice), so one lobe has area \(\frac{\pi}{2n} / 2n = \frac{\pi}{4n}\).
A hyperbola has an equation of the form

$$r = \frac{1}{1 - \epsilon \cos \theta}$$

when \( \epsilon > 1 \).

The two "bad angles" occur when \( 1 - \epsilon \cos \theta = 0 \), i.e. \( \cos \theta = \frac{1}{\epsilon} \).

These angles are the rays parallel to the asymptotes. Hence

$$\cos \left( \pm \frac{\pi}{4} \right) = \frac{1}{\epsilon}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{1}{\epsilon}$$

$$\Rightarrow \quad \epsilon = \sqrt{2}$$

So one equation for such a hyperbola is

Rectangular: \( r \sqrt{2} \cos \theta = 1 \) \( \Rightarrow \quad r^2 = 2x^2 + 2 \sqrt{2} x + 1 \)

$$\Rightarrow \quad x^2 + 2 \sqrt{2} x + y^2 + 1 = 0$$

$$\Rightarrow \quad (x + \sqrt{2})^2 + y^2 = 1$$

(6) \( r = \frac{1}{A - \epsilon \cos \theta} = \frac{1}{A} \cdot \frac{1}{1 - \frac{1}{A} \epsilon \cos \theta} \), when \( A \neq 0 \), or \( (x + \sqrt{2})^2 + y^2 = 1 \).

The case \( A=0 \) is special; in that case \( r = \frac{1}{-\epsilon \cos \theta} \) gives \( x = -1 \); the graph is just a vertical line.

When \( A \neq 0 \), the curve is a conic section with eccentricity \( e = 1/A \). The three cases we discussed in class were \( e = 0 \) (which doesn't occur here), \( 0 < e < 1 \), \( e = 1 \), and \( e > 1 \). These correspond to \( A > 0 \), \( A = 0 \), and \( 0 < A < 1 \). When \( A \) is negative, the curve turns out to be the same as with \(-A\) (since \( r(\theta + \pi) = \frac{1}{A - \epsilon \cos(\theta + \pi)} = \frac{1}{A + \epsilon \cos \theta} = -\frac{1}{A - \epsilon \cos \theta} \)) so the shape depends only on \( |A| \).
So the following cases are possible:

- \( A = 0 \): line
  - \( x = 1 \)
- \( 0 < A < 1 \) or \(-1 < A < 0 \): hyperbola
  - \( \sin \left( \frac{1}{a} \right) > 1 \)
- \( A = \pm 1 \): parabola

A > 1 or A < -1: ellipse

\[ \text{(since } \frac{1}{a^2} < 1 \) \]

7. a) \( z^8 = 1 \) means \( r^8 \cdot e^{8i\theta} \) if \( z = r \cdot e^{i\theta} \).

So \( r = 1 \) and \( \theta \) (1,8\( \theta \)) must be the same point as \((1,0) = r(0)\).

The solutions are

\[ z = 1, e^{\frac{2\pi}{8}}, e^{\frac{4\pi}{8}}, ..., e^{\frac{14\pi}{8}} \]

\[ i.e. \quad z = 1, e^{\frac{i\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}, e^{i\frac{9\pi}{4}}. \]

or in rectangular form,

\[ z = \pm 1 \text{ or } \pm i \text{ or } \pm \frac{1}{2} \pm i \frac{1}{2}. \]

b) \( z^3 = 8i \) means \( r^3 \cdot e^{3i\theta} = 8 \cdot e^{i\frac{\pi}{2}} \).

So \( r = 2 \) and \( \theta \) is one of \( \frac{\pi}{6}, \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6} \), or \( \frac{\pi}{6} + \frac{4\pi}{3} = \frac{3\pi}{2} \).

\[ \text{in polar form, } z = 2e^{\frac{i\pi}{6}}, 2e^{i\frac{5\pi}{6}}, \text{ or } 2e^{i\frac{3\pi}{2}}. \]

in rectangular form, \[ z = \frac{\sqrt{3} + i}{2}, -\frac{\sqrt{3} + i}{2}, \text{ or } -2i. \]
8. Let \( z = a + bi \), so \( \bar{z} = a - bi \).

   a) \( z + \bar{z} = 2a \), so any real number is possible.

   b) \( z - \bar{z} = 2bi \), so any imaginary number is possible.

   c) \( z \cdot \bar{z} = a^2 + b^2 \), so any nonnegative real number is possible.

   d) \( \frac{z}{\bar{z}} = \frac{a + bi}{a - bi} = \frac{(a + bi)^2}{a^2 + b^2} = \frac{a^2 + 2abi - b^2}{a^2 + b^2} = \frac{1}{a^2 + b^2} \cdot (a^2 - b^2 + 2abi) \).

   or, in polar form, \( \frac{r e^{i\theta}}{r e^{-i\theta}} = \frac{z}{\bar{z}} = e^{2i\theta} \), which can be any complex number of absolute value 1 as \( \theta \) varies.

9. Begin with the second equation:
   \[(1 - i)z + (5 + i)w = 0\]

   \[\Rightarrow (1 - i)z = -(5 + i)w\]

   \[\Rightarrow z = \frac{5 + i}{1 - i} \cdot w = \frac{(5 + i)(1 + i)}{(1 - i)(1 + i)} \cdot w\]

   \[= \frac{5 + 5i + i - 1}{1 + 1} \cdot w = \frac{4 + 6i}{2} \cdot w = (-2 - 3i)w\]

   Substituting for \( z \), in the first equation:

   \[(1 + i)(-2 - 3i)w + (5 + i)w = 4 + 3i\]

   \[(-2 - 3i)w + (2 + i)w = 4 + 3i\]

   \[(1 - 5i)w + (2 + i)w = 4 + 3i\]

   \[(3 - 4i)w = 4 + 3i\]

   \[w = \frac{4 + 3i}{3 - 4i} = \frac{(4 + 3i)(3 + 4i)}{3^2 + 4^2} = \frac{12 + 16i + 9i - 12}{25} = i\]
So \( w = i \) \& \( z = (-2-3i)w = (-2-3i) \cdot i = 3-2i \)

\[ \text{ie. } \begin{align*} z &= 3-2i \\ w &= i \end{align*} \]

10. Begin by finding \( f' \) \& \( f'' \) in each case:

<table>
<thead>
<tr>
<th>case</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( e^{ix} )</td>
<td>( ixe^{ix} )</td>
<td>( 4e^{ix} )</td>
</tr>
<tr>
<td>B</td>
<td>( xe^{x} )</td>
<td>( e^{x}+xe^{x} )</td>
<td>( 2xe^{x} )</td>
</tr>
<tr>
<td>C</td>
<td>( e^{x}\cos x )</td>
<td>( -e^{x}\cos x-e^{x}\sin x )</td>
<td>( 2e^{x}\sin x )</td>
</tr>
<tr>
<td>D</td>
<td>( e^{x}\sin x )</td>
<td>( e^{x}\sin x+e^{x}\cos x )</td>
<td>( 2e^{x}\cos x )</td>
</tr>
</tbody>
</table>

Then compute each of the four expressions to see which is 0:

<table>
<thead>
<tr>
<th></th>
<th>I ( f''-2f'+f )</th>
<th>II ( f''+2f'-2f )</th>
<th>III ( f''-f'-2f )</th>
<th>IV ( f''-2f'+2f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( e^{ix} )</td>
<td>10xe^{ix}</td>
<td>( -2xe^{ix} )</td>
<td>( 2xe^{ix} )</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>( 4xe^{ix} )</td>
<td>( e^{ix} )</td>
<td>( xe^{ix} )</td>
</tr>
<tr>
<td>C</td>
<td>( 3e^{x}\cos x +4e^{x}\sin x )</td>
<td>( \Phi )</td>
<td>( -e^{ix} +3e^{ix} \sin x )</td>
<td>( 4e^{ix} \cos x +4e^{ix} \sin x )</td>
</tr>
<tr>
<td>D</td>
<td>( -e^{ix} \sin x )</td>
<td>( 4e^{ix} \sin x +4e^{ix} \cos x )</td>
<td>( -3e^{ix} \sin x +e^{ix} \cos x )</td>
<td>0</td>
</tr>
</tbody>
</table>

Therefore

A solves III
B solves I
C solves II
D solves IV
a) Changing the "12" appears to change the number of "lobes" of the curve.
   e.g. making it a "1" produces
   (rough sketch).
   Changing to non-integre values can create many folds/lobes cross-crossing each other.

b) Making the "2" larger tends to equalize the sizes of the various lobes. If "2" becomes "50," the butterfly looks like a flower.
   (roughly).
   On the other hand, smaller values exaggerate the wings; the front lobes begin to bulge out.

c) There are many possible answers. For example:

- Replacing $e^{\cos \theta}$ by $0.5e^{\cos \theta}$ or $0.1e^{\cos \theta}$ makes the lobes more balanced (more like a flower), while replacing it with $Ze^{\cos \theta}$ exaggerates the front lobes of the wings.

- Changing the exponent "5" shifts the layer but preserves the basic shape of the butterfly.

- Changing the "4" (e.g. to 20 or 6) dramatically changes the sizes & number of lobes (it no longer looks much like a butterfly).