P. Set 3 Solutions

1.

a) $\vec{r}(t) = \vec{r}(0) + t\cdot\vec{v}$

$= \boxed{(-20,40,10) + t\cdot(30,40,0)}$ or $(30t-20, 40t+40, 10)$.

b) dist. to frog $= |(10,40,0) - \vec{r}(t)|$

$= |(10,40,0) - (-20,40,10) - t\cdot(30,40,0)|$

$= |(30,0,-10) - t\cdot(30,40,0)|$

$= \sqrt{(30-30t)^2 + (-40t)^2 + (-10)^2}$

$= \sqrt{900t^2 - 1800t + 900 + 1600t^2 + 100}$

$= \sqrt{2500t^2 - 1800t + 1000}$

To minimize this, minimize the square $2500t^2 - 1800t + 1000$, then take the square root.

Method 1: take the derivative.

$0 = \frac{d}{dt}(2500t^2 - 1800t + 1000)$

$0 = 5000\cdot t - 1800$

$\Rightarrow \boxed{t = \frac{9}{25}}$

So min. distance is

$\sqrt{2500\cdot(\frac{9}{25})^2 - 1800\cdot\frac{9}{25} + 1000}$

$= \sqrt{324 - 648 + 1000}$

$= \sqrt{676}$

$= \boxed{26\text{ cm.}}$

Method 2: complete the square.

$2500t^2 - 1800t + 1000$

$= (50t - 18)^2 + 1000 - 18^2$

$= (50t-18)^2 + 676$

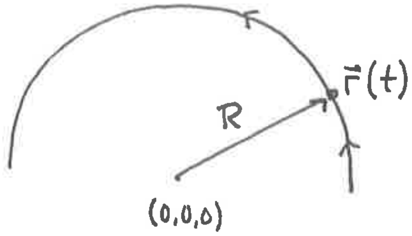
which is minimal when $50t = 18$, so $t = \frac{18}{50} = \boxed{\frac{9}{25}}$.

minimal distance is

$\sqrt{676} = \boxed{26\text{ cm.}}$

2

a)



$\vec{r}(t) = R\cdot(\cos(\omega t), \sin(\omega t), 0)$

$\vec{v}(t) = R\cdot\omega\cdot(-\sin(\omega t), \cos(\omega t), 0)$

speed $= |\vec{v}(t)| = R\omega$, so $\omega = \frac{250}{R}$.

So $\boxed{\vec{r}(t) = \cancel{25}\, R\cdot\left(\cos(\tfrac{250}{R}t), \sin(\tfrac{250}{R}t), 0\right)}$

b) $\vec{v}(t) = 250\cdot\left(-\sin(\tfrac{250}{R}t), \cos(\tfrac{250}{R}t), 0\right)$

$\vec{a}(t) = -\frac{250^2}{R}\cdot\left(\cos(\tfrac{250}{R}t), \sin(\tfrac{250}{R}t), 0\right)$

$\Rightarrow$ g-force $\vec{f}(t) = \vec{a}(t) + g\cdot\hat{k}$

$= \boxed{\left(-\frac{250^2}{R}\cdot\cos(\tfrac{250}{R}t), -\frac{250^2}{R}\cdot\sin(\tfrac{250}{R}t), g\right)}.$

c) $\vec{f}(t)\cdot k = g$ (using coordinates)

$= |\vec{f}(t)|\cdot 1\cdot\cos\vartheta$

$\Rightarrow \cos\vartheta = \frac{g}{|\vec{f}(t)|} = \frac{g}{\sqrt{(\frac{250^2}{R})^2 + g^2}}$ or $\vartheta = \cos^{-1}\left(\frac{g}{\sqrt{(\frac{250^2}{R})^2 + g^2}}\right)$.

d) If $\vartheta = 30°$, then $\cos\vartheta = \sqrt{3}/2$, so

$\frac{\sqrt{3}}{2} = \frac{g}{\sqrt{(\frac{250^2}{R})^2 + g^2}} \Rightarrow \frac{3}{4} = \frac{g^2}{(\frac{250^2}{R})^2 + g^2}$

$\Rightarrow 3\left[(\tfrac{250^2}{R})^2 + g^2\right] = 4g^2$

$\Rightarrow 3\cdot(\tfrac{250^2}{R})^2 = (4-3)g^2 = g^2$

$\Rightarrow \frac{250^2}{R} = \frac{g}{\sqrt{3}} \Rightarrow \underline{\frac{250^2\cdot\sqrt{3}}{g} = R}$

Therefore the time to make a $180°$ turn is

$$\frac{\text{distance}}{\text{speed}} = \frac{\pi \cdot R}{250}$$

$$= \boxed{\frac{250\sqrt{3}\cdot\pi \ m/s}{g}}$$

using $g = 9.8 \frac{m}{s^2}$, this gives $\frac{250 \cdot 1.732 \cdot \pi}{9.8} \approx$ $\boxed{138.8 \text{ seconds.}}$

3

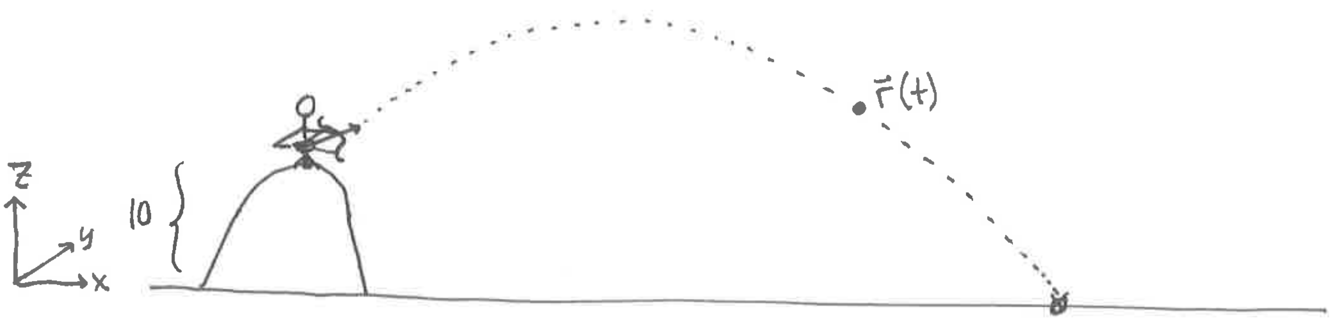
a) $\vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}_0 - \frac{1}{2} g t^2 \cdot \hat{k}$

$\vec{v}(t) = 0 + 1 \cdot \vec{v}_0 - g \cdot t \cdot \hat{k}$

$\boxed{\vec{v}(t) = \vec{v}_0 - g \cdot t \cdot \hat{k}}$

$\boxed{\vec{a}(t) = -g \cdot \hat{k}}$

b) $\vec{r}(t) = (0,0,10) + t \cdot (30,0,40) - \frac{1}{2} g t^2 \cdot (0,0,1)$



$\vec{r}(t) = (30t, 0, 10 + 40t - \frac{1}{2} g t^2)$

This hits the ground when $z = 0$, i.e. when

$10 + 40t - \frac{1}{2} g t^2 = 0.$

By the quadratic equation, this gives

$$t = \frac{-40 \pm \sqrt{40^2 + 4\cdot\frac{1}{2}g\cdot 10}}{-g}$$

$$t = \frac{1}{g}\cdot\left(40 \pm \sqrt{1600+20g}\right);$$ and the $\pm$ must be a $+$, otherwise t would be negative.

Therefore the arrow meets the ground at

$$\boxed{\vec{r}(t) = \left(\frac{30}{g}\cdot\left(40+\sqrt{1600+20g}\right),\ 0,\ 0\right)}$$

Assuming $\underline{g=10}$ m/s^2 this is

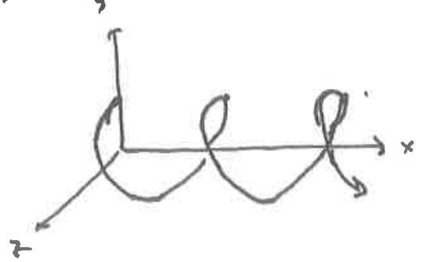
$$\left(\frac{30}{10}\cdot(40+\sqrt{1800}), 0, 0\right)$$
$$= (3\cdot(40+30\sqrt{2}), 0, 0)$$
$$= (120+90\sqrt{2}, 0, 0)$$
$$\approx (247 \text{ meters}, 0, 0)$$

or assuming $\underline{g=9.8}$ m/s^2, this is

$$\approx (252.181, 0, 0) \text{ meters.}$$

(4) $\vec{r}(t) = (12t^2, 5\cos(\pi t^2), 5\sin(\pi t^2))$

a) This is a <u>helix</u> (circle in the yz plane, moving along the x axis).



b)

$$\vec{v}(t) = (24t,\ -5\sin(\pi t^2)\cdot 2\pi t,\ \ 5\cos(\pi t^2)\cdot 2\pi t)$$

$$\boxed{\vec{v}(t) = (24t,\ -10\pi t\cdot \sin(\pi t^2),\ \ 10\pi t\cos(\pi t^2))}$$

$$\text{speed} = |\vec{v}(t)| = \sqrt{(24t)^2 + (10\pi t)^2\cdot[\sin^2(\pi t^2) + \cos^2(\pi t^2)]}$$

$$= \sqrt{(24t)^2 + (10\pi t)^2}$$

$$\boxed{|\vec{v}(t)| = t\cdot\sqrt{24^2 + 100\pi^2}}$$

$$\text{accel. } \vec{a}(t) = (24,\ -10\pi\sin(\pi t^2) - 10\pi t\cdot\cos(\pi t^2)\cdot 2\pi t,$$
$$10\pi\cos(\pi t^2) + 10\pi t\cdot(-\sin(\pi t^2))\cdot 2\pi t)$$

$$\boxed{\begin{aligned}\vec{a}(t) = (24, &-10\pi\sin(\pi t^2) - 20\pi^2 t^2\cos(\pi t^2),\\ &10\pi\cos(\pi t^2) - 20\pi^2 t^2\sin(\pi t^2))\end{aligned}}$$

c) $$\text{length} = \int_0^{2} |\vec{v}(t)|\,dt = \int_0^2 t\sqrt{24^2+100\pi^2}\,dt$$

$$= \sqrt{24^2+100\pi^2}\cdot\left[\tfrac{1}{2}t^2\right]_0^2$$

$$= \boxed{2\sqrt{24^2+100\pi^2}}$$

$$= 4\sqrt{144+25\pi^2}$$

$$\approx \underline{79.069}$$

⑤ a) Write $\vec{r}(t) = (\cos t, \sin t, 0) + (t, 0, t)$. So this is circular motion parallel to the ~~in the~~ xy plane, around a point moving in the direction $(1, 0, 1)$. It looks something like a "slanted" helix:

(Wolfram alpha can draw a nice plot.)

b) $\boxed{\begin{aligned}\vec{v}(t) &= (1-\sin t,\ \cos t, 1) \\ \vec{a}(t) &= (-\cos t,\ -\sin t,\ 0)\end{aligned}}$

N.B.: this is the same acceleration as simple circular motion $(\cos t, \sin t)$. All that has changed is the "reference frame".

c) arc-length $= \int_0^{10} |\vec{v}(t)|\,dt$

$$= \int_0^{10} \sqrt{(1-\sin t)^2 + \cos^2 t + 1}\,dt$$

$$= \int_0^{10} \sqrt{1-2\sin t + \sin^2 t + \cos^2 t + 1}$$

$$= \int_0^{10} \sqrt{3-2\sin t}\,dt$$

≈ 15.689, according to a computer.

6 a) $(x,y) = (7\cos\frac{\pi}{3},\ 7\sin\frac{\pi}{3})$
$= (7\cdot\frac{1}{2},\ 7\cdot\frac{\sqrt{3}}{2}) = \boxed{(\frac{7}{2},\ \frac{7\sqrt{3}}{2})}$

b) $(x,y) = (11\cos 0,\ 11\sin 0) = \boxed{(11, 0)}$

c) $(x,y) = (28\cos\frac{\pi}{6},\ 28\sin\frac{\pi}{6}) = \boxed{(14\sqrt{3},\ 14)}$

d) $(x,y) = (2\cos\frac{\pi}{2},\ 2\sin\frac{\pi}{2}) = \boxed{(0, 2)}$

7

a) $r = 7\sqrt{2}$

$\vartheta = \tan^{-1} 1 = \pi/4$ (quadrant I)

$\boxed{(r,\vartheta) = (7\sqrt{2}, \pi/4)}$

b) $r = \sqrt{5^2+12^2} = 13$

$\vartheta = \tan^{-1}\left(\frac{12}{5}\right) \approx 1.176$ or $67.38°$ (quad. I)

$\boxed{(r,\vartheta) = (13, \tan^{-1}(\tfrac{12}{5}))}$

$\approx (13, 67.38°)$.

c) $r = 24$

$\vartheta = \cancel{\tan^{-1}\left(\frac{-24}{0}\right)} - \frac{\pi}{2}$ (along neg. y-axis)

$\boxed{(r,\vartheta) = (24, -\pi/2)}$

d) $r = \sqrt{(7\sqrt{3})^2+7^2} = 14$

$\vartheta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi$ (quadrant III)

$= \frac{\pi}{6} + \pi = \frac{7}{6}\pi$ or $-\frac{5}{6}\pi$

$\boxed{(r,\vartheta) = (14, -\tfrac{5}{6}\pi)}$

8 $x^2 - 2x + y^2 - 2y = 0$

$r^2 - 2x - 2y = 0$

$r^2 - 2r\cos\vartheta - 2r\sin\vartheta = 0$

$\Rightarrow$ ($r = 0$ or) $r - 2\cos\vartheta - 2\sin\vartheta = 0$

$\boxed{r = 2\cos\vartheta + 2\sin\vartheta}$

9

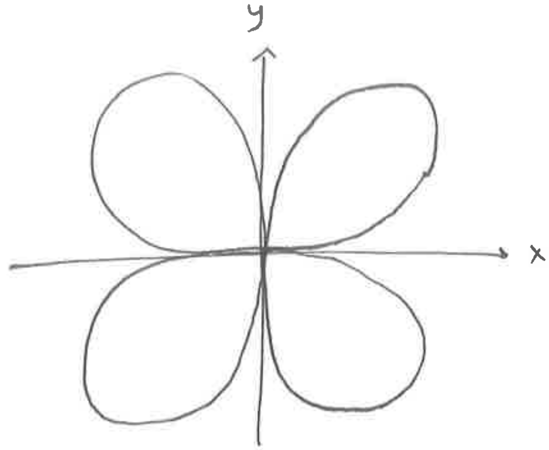
$r = 4\sin\theta\cos\theta$

$r^3 = 4 \cdot r\sin\theta \cdot r\cos\theta$

$(x^2+y^2)\cdot r = 4xy$

$\pm(x^2+y^2)^{3/2} = 4xy$

$$\boxed{(x^2+y^2)^3 = 16x^2y^2}$$



10

a) By facts about basic waves,

$$A_n = \frac{1}{\pi}\int_0^{2\pi} \sin(7x)\cos(nx)\,dx$$

$$= 0 \text{ for } \underline{\text{all}}\ n$$

and $$B_n = \frac{1}{\pi}\int_0^{2\pi} \sin(7x)\,\overset{\sin}{\cancel{\cos}}(nx)\,dx$$

$$= \begin{cases} 1 & \text{for } n=7 \\ 0 & \text{else} \end{cases}$$

~~then~~

so $\boxed{B_7 = 1, \text{ all other } A_n \text{ and } B_n \text{ are } 0}$.

b)

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_0^{2\pi} = \frac{1}{\pi}\cdot\left[\frac{1}{2}\cdot 4\pi^2 - 0\right] = 2\pi.$$

For any other n, use int. by parts

$$A_n = \int_0^{2\pi} x\cdot\cos(nx)\,dx \qquad \begin{array}{l} u = x \\ du = dx \end{array} \qquad \begin{array}{l} dv = \cos(nx)\,dx \\ v = \frac{1}{n}\sin(nx) \end{array}$$

$$= \left[\frac{x}{n}\cdot\sin(nx)\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{n}\sin(nx)\,dx$$

$$= 0 - \left[-\frac{1}{n^2}\cos(nx)\right]_0^{2\pi}$$

$$= 0.$$

Similarly, for any $n \neq 0$,

$$B_n = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin(nx)\,dx \qquad \begin{array}{l} u = x \\ du = dx \end{array} \qquad \begin{array}{l} dv = \sin(nx)\,dx \\ v = -\frac{1}{n}\cos(nx) \end{array}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cdot\cos(nx)\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}\left(-\frac{1}{n}\right)\cos(nx)\,dx$$

$$= -\frac{1}{\pi}\frac{1}{n}\cdot 2\pi\cdot 1 + \frac{1}{n}\cdot 0\cdot 1 + \frac{1}{\pi}\left[\frac{1}{n^2}\sin(nx)\right]_0^{2\pi}$$

$$= -\frac{2}{n}.$$

Therefore in particular:

$$\boxed{\begin{array}{ll} A_0 = 2\pi & B_1 = -2 \\ A_1 = 0 & B_2 = -1 \\ A_2 = 0 & \end{array}}$$