P. Set 2 Solutions

① $F(x) = kx$

$F(3)=25 \Rightarrow k = \frac{25}{3}$.

Thus $W = \int_3^5 F(x)\,dx = \int_3^5 \frac{25}{3}x\,dx = \left[\frac{25}{6}x^2\right]_3^5 = \frac{25}{6}\cdot(25-9)$

$= \boxed{\frac{200}{3}\ J} \approx 66.667\ J$.

② $\left.\begin{array}{l} F(1) = 5 \\ F(\frac{1}{2}) = 20 \end{array}\right\} \Rightarrow \left.\begin{array}{l} k\cdot 1 = 5 \\ k\cdot(\frac{1}{2})^{\gamma} = 20 \end{array}\right\} \Rightarrow$ $k=5$ and $\gamma = 2$.

$W = \int_2^3 (k\cdot x^{-\gamma})\,dx = \int_2^3 5\cdot x^{-2}\,dx = \left[-5x^{-1}\right]_2^3$

$= -5/3 + 5/2 = \boxed{5/6\ J} \approx 0.833\ J$.

NOTE: this is the work done by the gas (hence the work you can do with the piston). The work done by a force in the other direction would be $-5/6$ J. Either 5/6 or $-5/6$ will be marked correct; the problem was not worded well.

③ a) $P(t)V(t) = nR\cdot T(t)$

~~$\Rightarrow A\cdot x(t)\cdot\frac{x(t)}{A} = nR\cdot T(t)$~~

since $P(t) = F(t)/A = \frac{k}{A}\cdot x(t)^{-\gamma}$

and $V(t) = A\cdot x(t)$

this gives

$$\frac{k}{A}\cdot x(t)^{-\gamma}\cdot A\cdot x(t) = nR\cdot T(t)$$

$$\Rightarrow \boxed{\frac{k}{nR}\cdot x(t)^{-\gamma+1} = T(t)}$$

or $\frac{5}{nR}\cdot x(t)^{-1} = T(t)$ using values from #2.

b) $T(b)-T(a) = \frac{k}{nR}\cdot\left(x(b)^{-\gamma+1} - x(a)^{-\gamma+1}\right)$

while on the other hand, the work done by the gas is

$$W = \int_{x(a)}^{x(b)} F(x)\,dx = \int_{x(a)}^{x(b)} k\cdot x^{-\gamma}\,dx$$

$$= \left[\frac{k}{-\gamma+1}\cdot x^{-\gamma+1}\right]_{x(a)}^{x(b)}$$

$$= \frac{k}{-\gamma+1}\left[x(b)^{-\gamma+1} - x(a)^{-\gamma+1}\right]$$

hence $\frac{T(b)-T(a)}{W} = \frac{-\gamma+1}{nR}$, which is constant.

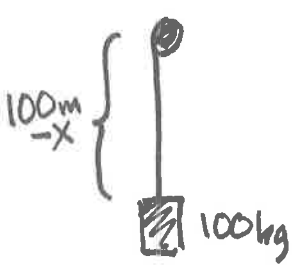
So temperature change is directly proportional to work.

(4)

a) $\int_1^b k\cdot x^{-1}\,dx = \left[k\ln|x|\right]_1^b = k\ln(b)$.

b) $\lim_{b\to\infty} k\ln(b) = \infty$. This means that the gas does an infinite amount of work as it expands, which is not physically plausible.

(5) After x meters of cable have been lifted up. $100-x$ meters of cable remain, so the tension in the cable is



$$F(x) = \Big[\underbrace{(100-x)\cdot 0.4}_{\text{mass of cable}} + \underbrace{100}_{\text{mass of weight}}\Big]\cdot g$$

so the total work necessary is

$$\int_0^{100} \left[(100-x)\cdot 0.4 + 100\right]\cdot g\,dx$$

$$= g\cdot\int_0^{100}\left(140-\tfrac{2}{5}x\right)dx$$

$$= g\cdot\left[140x-\tfrac{1}{5}x^2\right]_0^{100}$$

$$= g\cdot\left[14000-2000\right]$$

$$= \boxed{12{,}000\cdot g}$$

$$\approx 12{,}000\cdot 9.8 = \boxed{117{,}600\ \mathrm{J}}$$

assuming $g\approx 9.8\,\mathrm{m/s^2}$.

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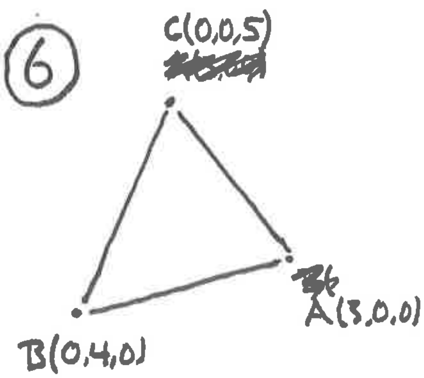


$$\vec{AB} = \vec{B}-\vec{A} = (-3,4,0)$$
$$\vec{BC} = \vec{C}-\vec{B} = (0,-4,5)$$
$$\vec{CA} = \vec{A}-\vec{C} = (3,0,-5)$$

sidelengths: $|\vec{AB}| = \sqrt{3^2+4^2+0^2} = 5$

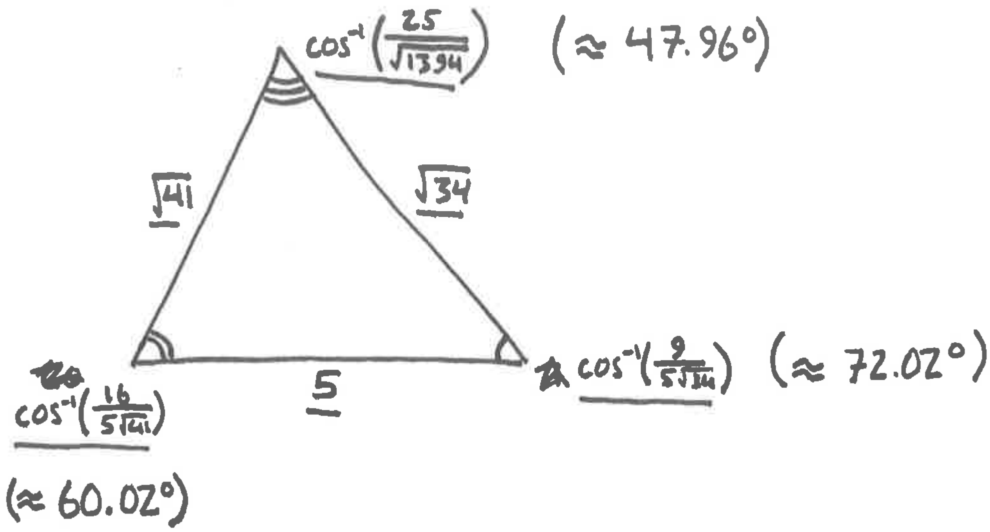
$$|\vec{BC}| = \sqrt{0^2+(-4)^2+5^2} = \sqrt{41}$$
$$|\vec{CA}| = \sqrt{3^2+0^2+(-5)^2} = \sqrt{34}$$

angles: $\cos(\angle CAB) = \dfrac{\vec{AC}\cdot\vec{AB}}{|\vec{AC}|\cdot|\vec{AB}|} = \dfrac{(-3,0,5)\cdot(-3,4,0)}{5\cdot\sqrt{34}} = \dfrac{9}{5\sqrt{34}}$

$$\cos(\angle ABC) = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}|\cdot|\vec{BC}|} = \frac{(3,-4,0)\cdot(0,-4,5)}{5\sqrt{41}} = \frac{16}{5\sqrt{41}}$$

$$\cos(\angle BCA) = \frac{\vec{CB}\cdot\vec{CA}}{|\vec{CB}|\cdot|\vec{CA}|} = \frac{(0,4,-5)\cdot(3,0,-5)}{\sqrt{34\cdot 41}} = \frac{25}{\sqrt{1394}}$$

so the sides & angles are as shown.



7. We know that

$$\frac{\vec{v}}{|\vec{v}|} = \left(\frac{12}{13}, \frac{4}{13}, \frac{3}{13}\right) \quad \text{and} \quad \vec{v}\cdot(0,0,1) = 3 \text{ km}.$$

Therefore $\boxed{|\vec{v}| = 13 \text{ km}}$ and $\vec{v} = (12, 4, 3)$.

8. Plane's position is $\vec{p}(t) = (0,2,0) + t\cdot\vec{v}$, where

$|\vec{v}| = 0.2 \ \frac{\text{km}}{\text{sec}}$ and $\vec{v}$ points from $(0,2,0)$ to $(0,14,5)$.

Thus

$$\frac{\vec{v}}{|\vec{v}|} = \frac{(0,14,5)-(0,2,0)}{|(0,14,5)-(0,2,0)|} = \frac{(0,12,5)}{|(0,12,5)|} = \left(0, \frac{12}{13}, \frac{5}{13}\right)$$

$$\Rightarrow \vec{v} = 0.2\cdot\left(0, \frac{12}{13}, \frac{5}{13}\right) = \left(0, \frac{12}{65}, \frac{5}{65}\right).$$

$$\Rightarrow \vec{p}(t) = \left(0, 2+\frac{12}{65}t, \frac{5}{65}t\right) \quad \text{kilometers.}$$

Thus the distance to the tower is

$$|\vec{p}(t) - (-\tfrac{1}{2}, 0, \tfrac{1}{5})|$$

$$= \left| \left(\tfrac{1}{2},\ 2 + \tfrac{12}{65}t,\ \tfrac{5}{65}t - \tfrac{1}{5}\right) \right|$$

$$= \boxed{\sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{12}{65}t + 2\right)^2 + \left(\tfrac{5}{65}t - \tfrac{1}{5}\right)^2}}$$

$$\approx \sqrt{0.25 + (0.185t + 2)^2 + (0.0769t - 0.2)^2}$$

(9) $\vec{F}_1 = \lambda \cdot \vec{A}$ for some λ,

* and $\vec{A} \cdot (\vec{F}) = 1 \cdot 3 - 1 \cdot 2 + 2 \cdot 0 = 1$

$$= \vec{A} \cdot (\vec{F}_1 + \vec{F}_2) = \vec{A} \cdot (\lambda \vec{A}) + \cancel{\vec{A} \cdot \vec{F}_2} \to 0$$

$$= \lambda \cdot \vec{A} \cdot \vec{A} = \lambda (1^2 + (-1)^2 + 2^2) = 6\lambda$$

hence $\lambda = 1/6$. This means

$$\boxed{\begin{aligned} &\vec{F}_1 = \tfrac{1}{6}\vec{A} = \left(\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{1}{3}\right) \\ \text{and}\quad &\vec{F}_2 = \vec{F} - \vec{F}_1 = \left(\tfrac{17}{6}, \tfrac{13}{6}, -\tfrac{1}{3}\right) \end{aligned}}$$

(10)

$$\int_{1/2}^{\sqrt{3}/2} \sin^{-1}(x)\,dx \qquad u=\sin^{-1}(x) \quad dv=dx \qquad du=\frac{1}{\sqrt{1-x^2}}\,dx \quad v=x$$

$$= \left[x\cdot\sin^{-1}(x)\right]_{1/2}^{\sqrt{3}/2} - \int_{1/2}^{\sqrt{3}/2} \frac{x}{\sqrt{1-x^2}}\,dx \qquad \begin{matrix} w=1-x^2 \\ dw=-2x\,dx \end{matrix}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\pi}{6} - \int_{3/4}^{1/4} \frac{(-1/2)\,dw}{\sqrt{w}}$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} - \int_{1/4}^{3/4} \frac{dw}{2\sqrt{w}} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} - \left[\sqrt{w}\right]_{1/4}^{3/4}$$

$$= \boxed{\frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} - \frac{\sqrt{3}}{2} + \frac{1}{2}} \approx \boxed{0.279}$$

(11) Let $f(x) = A\cdot\sin(2x) + B\cdot\cos(x)$. Then by the same reasoning as in Pset1, problem 10,

$$\int_0^{2\pi} f(x)\sin(2x)\,dx = \pi\cdot A$$

$$\int_0^{2\pi} f(x)\cos x\,dx = \pi\cdot B$$

$$\int_0^{2\pi} f(x)^2\,dx = \pi\cdot(A^2+B^2).$$

Letting $A=3$, $B=4$ will meet the desired conditions.

$$\boxed{f(x) = 3\sin(2x) + 4\cos x}$$