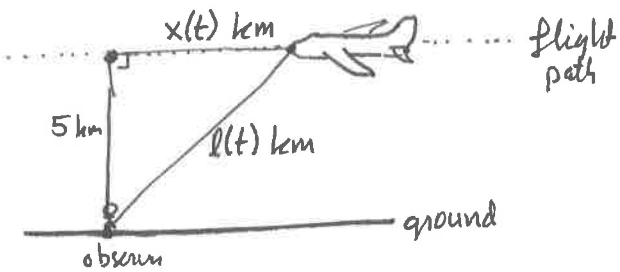
# P. Set 1 Solutions

① Avg. value $= \frac{1}{\pi - 0} \cdot \int_0^{\pi} \sin\theta \, d\theta = \frac{1}{\pi} \cdot \left[-\cos\theta\right]_0^{-\pi} = \frac{-(-1)-(-1)}{\pi} = \boxed{\frac{2}{\pi}}$

②




Define functions $x(t)$ & $\ell(t)$ as shown, where $t$ is in seconds and $t=0$ is the current moment.

We know: $\ell(0) = 13$, $\ell'(t) = -0.2$ ($-200 \frac{m}{s} = -0.2 \frac{km}{s}$).

By the Pythagorean theorem:

$$x(t)^2 + 5^2 = \ell(t)^2 \qquad \text{so} \qquad x(0) = 12.$$

$$\Rightarrow \quad 2x(t)\cdot x'(t) + 0 = 2\cdot \ell(t)\cdot \ell'(t)$$

$$\Rightarrow \quad x'(t) = \ell(t)\cdot \ell'(t) / x(t)$$

$$\Rightarrow \quad x'(0) = \ell(0)\cdot \ell'(0)/x(0)$$

$$= 13\cdot(-0.2)/12 = -\frac{13}{60}$$

so the plane's speed is $\boxed{\frac{13}{60} \text{ km/sec}}$, or $\boxed{780 \text{ km/hr}}$.

③ a) $\int_0^{\ln 2} \frac{e^{3x}}{1+e^{6x}}\,dx \qquad u = e^{3x}, \; du = 3e^{3x}dx$

$$= \int_1^8 \frac{\frac{1}{3}du}{1+u^2} = \frac{1}{3}\cdot\left[\arctan x\right]_1^8 = \boxed{\frac{1}{3}\cdot\arctan(8) - \frac{\pi}{12}} \approx \boxed{0.220}$$

b) $\int_2^3 \frac{dx}{x\cdot \ln x} \qquad u = \ln x, \; du = \frac{1}{x}dx$

$$= \int_{\ln 2}^{\ln 3} \frac{du}{u} = \left[\ln|u|\right]_{\ln 2}^{\ln 3} = \boxed{\ln(\ln 3) - \ln(\ln 2)} \approx \boxed{0.461}$$

c) $\int_0^1 x(1-x)^{2/3}dx \quad u=1-x \quad du=-dx$

$$= -\int_1^0 (1-u)\cdot u^{2/3}du = \int_0^1 (1-u)\cdot u^{2/3}du = \int_0^1 (u^{2/3} - u^{5/3})du$$

$$= \left[\frac{3}{5}u^{5/3} - \frac{3}{8}\cdot u^{8/3}\right]_0^1 = \frac{3}{5} - \frac{3}{8} = \boxed{9/40} = \boxed{0.225}.$$

d) $\int x\cdot e^{-x^2/2}dx \quad u=-x^2/2 \quad du=-x$

$$= -\int e^u du = \boxed{-e^{-x^2/2}+C}$$

(4) Profit = (p-5) dollars/widget $\cdot 2^{-p}$ million widgets/year

$f(p) = 2^{-p}\cdot(p-5)$ million dollars per year.

Then $f'(p) = (-\ln 2)\cdot 2^{-p}\cdot(p-5) + 2^{-p}\cdot 1$

$$= 2^{-p}\left(1 - \ln 2\cdot(p-5)\right)$$

Hence $f'(p)=0 \iff 0 = 1-\ln 2\cdot(p-5)$

$\iff p = 5+\frac{1}{\ln 2}$.

Since $f'(p)>0$ for smaller p and $f'(p)<0$ for larger p, this is the global maximum. So the best price is $\boxed{5+\frac{1}{\ln 2}\text{ dollars}} \approx \boxed{\$6.44}$

(5) Let $f(\vartheta) = (\sin\vartheta)^{\tan^2\vartheta}$. Then

$$\ln f(\vartheta) = \tan^2\vartheta\cdot\ln(\sin\vartheta)$$

$$\Rightarrow \lim_{\vartheta\to\pi/2}\ln f(\vartheta) = \lim_{\vartheta\to\pi/2}\frac{\ln(\sin\vartheta)}{\cot^2\vartheta}$$ , which is indeterminate form 0/0

L'hôpital: $$= \lim_{\vartheta\to\pi/2}\frac{\cos\vartheta/\sin\vartheta}{2\cdot\cot\vartheta\cdot(-\csc^2\vartheta)} = \lim_{\vartheta\to\pi/2}\left(-\frac{\cos\vartheta/\sin\vartheta}{2\cdot\frac{\cos\vartheta}{\sin\vartheta}\cdot\frac{1}{\sin^2\vartheta}}\right)$$

$$= \lim_{\vartheta\to\pi/2}\left(-\frac{\sin^2\vartheta}{2}\right) = -1/2.$$

Therefore $\lim_{\vartheta\to\pi/2} f(\vartheta) = e^{\lim_{\vartheta\to\pi/2}\ln(f(\vartheta))} = e^{-1/2} = \boxed{\frac{1}{\sqrt{e}}} \approx \boxed{0.607}$

6

a) $\int_0^{\pi} x\cdot \sin x\,dx$ $\quad u=x \quad dv=\sin x\,dx$ $\quad du=dx \quad v=-\cos x$

$=\left[-x\cos x\right]_0^{\pi}-\int_0^{\pi}(-\cos x)\,dx$

$=(-\pi)\cdot(-1)-0+\left[\sin x\right]_0^{\pi}$

$=\boxed{\pi}$.

b) $\int_{-\pi/2}^{\pi/2} x^2\cos x\,dx$ $\quad u=x^2 \quad dv=\cos x\,dx$ $\quad du=2x\,dx \quad v=\sin x$

$=\left[x^2\sin x\right]_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}2x\sin x\,dx$ $\quad u=2x \quad v=\sin x\,dx$ $\quad du=2dx \quad v=-\cos x$

$=\left(\frac{\pi}{2}\right)^2\cdot\left(1-(-1)\right)-\left[(2x)(-\cos x)\right]_{-\pi/2}^{\pi/2}+\int_{-\pi/2}^{\pi/2}2\cdot(-\cos x)\,dx$

$=\frac{\pi^2}{2}-\left[\pi\cdot(0-0)\right]+\left[-2\sin x\right]_{-\pi/2}^{\pi/2}$

$=\frac{\pi^2}{2}+\left[-2-(2)\right]=\boxed{\frac{\pi^2}{2}-4}\approx\boxed{0.935}$

c) $\int x^2\cdot 3^x\,dx$ $\quad u=x^2 \quad dv=3^x dx$ $\quad du=2x\,dx \quad v=\frac{1}{\ln 3}\cdot 3^x$

$=\frac{1}{\ln 3}\cdot x^2\cdot 3^x-\frac{2}{\ln 3}\int x\cdot 3^x\,dx$ $\quad u=x \quad dv=3^x dx$ $\quad du=dx \quad v=\frac{1}{\ln 3}\cdot 3^x$

$=\frac{1}{\ln 3}\cdot x^2\cdot 3^x-\frac{2}{(\ln 3)^2}\cdot x\cdot 3^x+\frac{2}{(\ln 3)^2}\int 3^x\,dx$

$=\boxed{\frac{1}{\ln 3}x^2\cdot 3^x-\frac{2}{(\ln 3)^2}\cdot x\cdot 3^x+\frac{2}{(\ln 3)^3}\cdot 3^x+C}$

d) $\int_1^{10} x\cdot\ln x\,dx$ $\quad u=\ln x \quad dv=x\,dx$ $\quad du=\frac{1}{x}dx \quad v=\frac{1}{2}x^2$

$=\left[\frac{1}{2}x^2\ln x\right]_1^{10}-\int_1^{10}\frac{1}{2}x^2\cdot\frac{1}{x}\,dx$

$=\frac{1}{2}\cdot 10^2\ln 10-\left[\frac{1}{4}x^2\right]_1^{10}=\boxed{50\ln 10-\frac{99}{4}}\approx\boxed{90.379}$

(7) $\int 5^x \sin(x/5)\,dx$ $\qquad u=5^x \quad dv=\sin(x/5)dx$
$\qquad du=\ln 5\cdot 5^x \quad v=-5\cos(x/5)$

$= -5^{x+1}\cos(x/5) + \int 5\ln 5\cdot 5^x \cos(x/5)$ $\qquad u=5^x \quad dv=\cos(x/5)$
$\qquad du=\ln 5\cdot 5^x \quad v=+5\sin(x/5)$

$= -5^{x+1}\cos(x/5) + 5\ln 5\cdot 5^x\cdot 5\sin(x/5) - \int 5^2(\ln 5)^2\cdot 5^x \text{cosin}(x/5)\,dx$

$\Rightarrow \left(1+25(\ln 5)^2\right)\cdot\int 5^x\sin(x/5)\,dx = 25\ln 5\cdot 5^x\sin x - 5\cdot 5^x\cos(x/5) + C$

$\Rightarrow \int 5^x \sin(x/5)\,dx = \boxed{\dfrac{25\ln 5}{1+25(\ln 5)^2}\cdot 5^x\sin(x/5) - \dfrac{5}{1+25(\ln 5)^2}\cdot 5^x\cos(x/5) + C}$

(8) $\int_0^{\pi^2}\sin(\sqrt{x})\,dx$ $\qquad u=\sqrt{x}$
$\qquad du=\frac{1}{2\sqrt{x}}\,dx = \frac{1}{2u}\,dx$

$= \int_0^{\pi} 2u\cdot\sin(u)\,du = \boxed{2\pi}$ (using the result of problem 6a).

(9) a) $\int_0^{\pi/2}\sin x\cos^5 x\,dx$ $\qquad u=\cos x$
$\qquad du=-\sin x\,dx$

$= \int_1^0 (-u^5)\,du = \left[\frac{1}{6}u^6\right]_0^1 = \boxed{1/6}$

b) $\int\sqrt{\cos x}\,\sin x\,dx$ $\qquad u=\cos x$
$\qquad du=-\sin x\,dx$

$= -\int\sqrt{u}\,du = -\frac{2}{3}u^{3/2}+C = \boxed{-\frac{2}{3}(\cos x)^{3/2}+C}$

c) $\int_0^{4\pi}\sin^2 x\cos^2 x\,dx$

$= \int_0^{4\pi}\left(\frac{1}{2}(1-\cos(2x))\right)\left(\frac{1}{2}(1+\cos(2x))\right)dx = \frac{1}{4}\int_0^{4\pi}\left(1-\cos^2(2x)\right)dx$

$= \frac{1}{4}\cdot[x]_0^{4\pi} - \frac{1}{4}\int_0^{4\pi}\cos^2(2x)\,dx$

$= \pi - \frac{1}{4}\int_0^{4\pi}\left(\frac{1}{2}(1+\cos(4x))\right)dx = \pi - \frac{1}{4}\left[\frac{1}{2}x\right]_0^{4\pi} - \frac{1}{8}\cdot\frac{1}{4}\cdot[\sin 4x]_0^{4\pi}$

$= \pi - \frac{1}{4}\cdot\frac{1}{2}\cdot 4\pi - \frac{1}{8}\cdot\frac{1}{4}\cdot 0 = \boxed{\frac{\pi}{2}}$

d) $\int_{-\pi/6}^{\pi/6} \cos^3 x\,dx = \int_{-\pi/6}^{\pi/6} (1-\sin^2 x)\cos x\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$= \int_{-1/2}^{1/2} (1-u^2)\,du = \left[u - \frac{1}{3}u^3\right]_{-1/2}^{1/2} = \left(\frac{1}{2} - \frac{1}{24}\right) - \left(-\frac{1}{2} + \frac{1}{24}\right)$

$= \cancel{\boxed{11/24}} = \boxed{11/12} \approx \boxed{0.917}$

e) $\int_{-\pi/3}^{\pi/3} \sec^4 x\,dx$ $\quad u = \tan x$, $du = \sec^2 x\,dx$

$= \int_{-\sqrt{3}}^{\sqrt{3}} (1+u^2)\,du = \left[u + \frac{1}{3}u^3\right]_{-\sqrt{3}}^{\sqrt{3}} = \left(\sqrt{3} + \frac{1}{3}\cdot 3\sqrt{3}\right) - \left(-\sqrt{3} - \frac{1}{3}\cdot 3\sqrt{3}\right)$

$= \boxed{4\sqrt{3}} \approx \boxed{6.928}$

f) $\int \tan^4 x\,dx = \int (\sec^2 x - 1)\cdot \tan^2 x\,dx$

$= \int \sec^2 x \cdot \tan^2 x\,dx - \int \tan^2 x\,dx$

$u = \tan x$ $du = \sec^2 x\,dx$

$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx$

$= \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$

(10) Use the facts:

$$\int_0^{2\pi} \sin(px)\cos(qx)\,dx = 0$$

$$\int_0^{2\pi} \sin(px)\sin(qx)\,dx = \begin{cases} 0 & p \neq q \\ \pi & p = q \end{cases}$$

$$\int_0^{2\pi} \cos(px)\cos(qx)\,dx = \begin{cases} 0 & p \neq q \\ \pi & p = q \end{cases}$$

a) $\int_0^{2\pi} f(x)\,dx = A\cdot\int_0^{2\pi} dx + B\int_0^{2\pi} \sin x\,dx + C\cdot\int_0^{2\pi} \sin(2x)\,dx$

$= \boxed{2\pi\cdot A}$

b) $\int_0^{2\pi} f(x)\sin x\,dx = A\cdot\int_0^{2\pi} \sin x\,dx + B\cdot\int_0^{2\pi} \sin^2 x\,dx + C\cdot\int_0^{2\pi} \sin x\cdot\sin(2x)\,dx$

$= \boxed{\pi\cdot B}$

c) $\int_0^{2\pi} f(x)\cos x\,dx = A\cdot\int_0^{2\pi}\cos x\,dx + B\int_0^{2\pi}\sin x\cos x\,dx + C\cdot\int_0^{2\pi}\sin(2x)\cos x\,dx$

$= \boxed{0}$.

d) $\int_0^{2\pi} f(x)\sin 2x\,dx = A\cdot\int_0^{2\pi}\sin(2x)dx + B\cdot\int_0^{2\pi}\sin x\sin(2x)dx + C\cdot\int_0^{2\pi}\sin^2(2x)dx$

$= \boxed{\pi\cdot C}$

e) $\int_0^{2\pi} f(x)\cos(2x)dx = A\cdot\int_0^{2\pi}\cos(2x)dx + B\int_0^{2\pi}\sin x\cos 2x\,dx + C\cdot\int_0^{2\pi}\sin(2x)\cos(2x)dx$

$= \boxed{0}$

f) $\int_0^{2\pi} f(x)^2dx = \int_0^{2\pi}(A+B\sin x+C\sin(2x))dx$

$= \cancel{\int_0} = A^2\int_0^{2\pi}1dx + 2AB\int_0^{2\pi}\sin x\,dx + 2AC\int_0^{2\pi}\sin(2x)dx$

$+ 2BC\int_0^{2\pi}\sin x\sin(2x)dx + B^2\int_0^{2\pi}\sin x\cdot\sin x + C^2\int_0^{2\pi}\sin(2x)\sin(2x)dx$

$= \boxed{2\pi\cdot A^2 + \pi\cdot B^2 + \pi\cdot C^2}$

11 a) $\int_{-1}^{1}\frac{dx}{\sqrt{9-x^2}}$ $\quad x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$ $\quad \theta = \sin^{-1}(x/3)$, $\sqrt{9-x^2} = 3\cos\theta$

$= \int_{-\sin^{-1}(1/3)}^{\sin^{-1}(1/3)}\frac{3\cos\theta\,d\theta}{3\cos\theta} = \left[\sin^{-1}(\tfrac{1}{3}) - (-\sin^{-1}(\tfrac{1}{3}))\right]$

$= \boxed{2\cdot\sin^{-1}(\tfrac{1}{3})} \approx \boxed{0.680}$

b) $\int\frac{dx}{\sqrt{x^2-3}}$ $\quad x = \sqrt{3}\cdot\sec\theta$, $dx = \sqrt{3}\sec\theta\cdot\tan\theta\,d\theta$ $\quad \sqrt{x^2-3} = \sqrt{3}\tan\theta$

$= \int\frac{\sqrt{3}\cdot\sec\theta\cdot\tan\theta\,d\theta}{\sqrt{3}\cdot\tan\theta} = \ln|\sec\theta+\tan\theta| + C$

$= \boxed{\ln\left|\tfrac{1}{\sqrt{3}}x + \tfrac{1}{\sqrt{3}}\cdot\sqrt{x^2-3}\right| + C}$

$= \ln|x+\sqrt{x^2-3}| + C$ $\quad$ (since $\ln(\tfrac{1}{\sqrt{3}}f(x)) = \ln f(x) + \ln\tfrac{1}{\sqrt{3}}$).

c) $\int_{-1}^{1} \frac{dx}{(1+x^2)^2}$

$x = \tan\vartheta$
$1+x^2 = \sec^2\vartheta$
$dx = \sec^2\vartheta\, d\vartheta$

$$= \int_{-\pi/4}^{\pi/4} \frac{\sec^2\vartheta\, d\vartheta}{\sec^4\vartheta} = \int_{-\pi/4}^{\pi/4} \cos^2\vartheta\, d\vartheta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos(2\vartheta))\, d\vartheta$$

$$= \left[\frac{1}{2}\vartheta + \frac{1}{4}\sin(2\vartheta)\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{4} + \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{4} = \boxed{\frac{\pi}{4} + \frac{1}{2}} \approx \boxed{1.285}$$

d) $\int_{\sqrt{2}}^{2} \frac{(x^2-2)^{3/2}}{x} dx$

$x = \sqrt{2}\cdot\sec\vartheta$
$x^2 - 2 = 2\cdot\tan^2\vartheta$
$dx = \sqrt{2}\cdot\sec\vartheta\cdot\tan\vartheta\, d\vartheta$

$\vartheta = \sec^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$$= \int_0^{\pi/4} \frac{2\sqrt{2}\cdot\tan^3\vartheta}{\sqrt{2}\cdot\sec\vartheta}\cdot\sqrt{2}\sec\vartheta\cdot\tan\vartheta\, d\vartheta = \int_0^{\pi/4} \tan^4\vartheta\, d\vartheta \cdot 2\sqrt{2}$$

$$= 2\sqrt{2}\cdot\left[\frac{1}{3}\tan^3\vartheta - \tan\vartheta + \vartheta\right]_0^{\pi/4}$$

(using the answer to problem 9 f)

$$= 2\sqrt{2}\cdot\left[\frac{1}{3}\cdot 1^3 - 1 + \frac{\pi}{4}\right] = 2\sqrt{2}\cdot\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$= \boxed{\frac{\sqrt{2}}{2}\cdot\pi - \frac{4\sqrt{2}}{3}} \approx \boxed{0.336}$$

12 $\int \frac{dx}{x^2-2x+2} = \int \frac{dx}{(x-1)^2+1} = \boxed{\tan^{-1}(x-1) + C}$