1. Consider the following function, which has period 2. This function is an example of a function called a "sawtooth wave."

$$\begin{aligned} f(x) &= x \qquad \text{when } -1 \le x < 1 \\ f(x+2) &= f(x) \qquad \text{for all } x \end{aligned}$$

Find the (period 2) real Fourier series of $f(x)$.

Note. For the remainder of this problem set, all Fourier series will have period 2π, in order to keep the notation simple. This will also be the case on all exam problems about Fourier series.

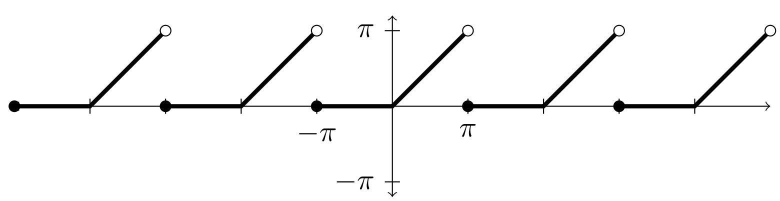
2. Consider the periodic function $f(x)$ defined by $f(x) = \pi^2 - x^2$ for x in $[-\pi, \pi]$, and $f(x+2\pi) = f(x)$ for all x.

   (a) Sketch the graph of this function for x in $[-5\pi, 5\pi]$.

   (b) Compute the real Fourier series of $f(x)$.

   (c) Compute the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ by setting $x = 0$ in the Fourier series, and noting that the result is equal to $f(0)$.

3. Let $f(x)$ be the 2π-periodic piecewise-linear function depicted in the following graph.



Find the real Fourier series of $f(x)$.

4. Convert each (finite) real Fourier series to a (finite) complex Fourier series.

   (a) $5 + 2\sin x + 3\cos(2x)$

   (b) $1 - 4\cos x + 3\sin x$

   (c) $\frac{1}{2}\sin x + \frac{1}{4}\sin(2x) + \frac{1}{8}\sin(3x)$

   (d) $6\cos x + 2\sin x + 5\sin(2x) + 3\cos(3x)$

5. Convert each (finite) complex Fourier series to a (finite) real Fourier series. Simplify your answer enough that there are no imaginary numbers in any of the coefficients.

   (a) $5e^{-ix} + 5e^{ix}$

   (b) $(1+i)e^{-2ix} + (1-i)e^{2ix}$

   (c) $\frac{1}{1+2i}e^{-2ix} + \frac{1}{1+i}e^{-ix} + \frac{1}{1-i}e^{ix} + \frac{1}{1-2i}e^{2ix}$

   (d) $ie^{-3ix} - e^{-2ix} - ie^{-ix} + 1 + ie^{ix} - e^{2ix} - ie^{3ix}$

6. A circuit consisting of a 1 Henry inductor, a 2 Ohm resistor, and a 0.2 Farad capacitor is attached to a power source. If the voltage of the power source is given by a function $V(t)$, then the charge $Q(t)$ on the capacitor obeys the following differential equation.

$$Q''(t) + 2Q'(t) + 5Q(t) = V(t)$$

Assume that $V(t)$ is the function $V(t) = 2\cos t + 2\cos(2t) + 2\cos(3t)$. The goal of this problem is to find the steady-state solution $Q(t)$ (that is, the unique solution that is periodic with period 2π).

(a) Let $c_n(V)$ denote the complex Fourier coefficients of $V(t)$, and let $c_n(Q)$ denote the complex Fourier coefficients of $Q(t)$. Using the differential equation, express $c_n(Q)$ in terms of $c_n(V)$.

(b) Determine the complex Fourier coefficients $c_n(V)$ of $V(t)$. Use your answer from part (a) to determine the complex Fourier coefficients $c_n(Q)$ of $Q(t)$.

(c) Find the real Fourier series of $Q(t)$.

Note. On problem 9 of the previous problem set, you used a different type of infinite series (Taylor series) to solve a different sort of problem: approximating the value of $Q(t)$ for values of t near 0 for a similar differential equation. In contrast, the method in this problem provides a way to study the steady-state solution, which will describe the function $Q(t)$ in the *long term.*

7. Evaluate the sum $\sum_{n=1}^{\infty} n^2 x^n$ where it converges (as a function of x).

Hint. Apply problem 8 from problem set 9.

8. Consider the radioactive particle discussed in problem 5 of problem set 8. As in that problem, the probability that the particle will decay on day n is $p_n = 0.999^{n-1} \cdot 0.001$. In that problem, you computed the expected value of the day of decay, denoted μ, and found that it is equal to 1000 days.

Compute the *standard deviation* of the day of decay of the particle. This is defined to be the number σ, where

$$\sigma = \sqrt{\left(\sum_{n=1}^{\infty} n^2 \cdot p_n\right) - \mu^2}.$$

Hint. Use problem 7.

Note. The standard deviation measures roughly how much you can expect the true day of decay to deviate from the expected value. It is often defined by the equation $\sigma = \sqrt{\sum_{n=1}^{\infty} (n-\mu)^2 p_n}$ (you can convince yourself with a little algebra that this is equal to what is written above).