**POWER SERIES (→ TAYLOR SERIES)**

Power series: has a separable form \( c \) is the "center"

\[
a_0 + a_1 (x-c) + a_2 (x-c)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n
\]

To converge for some values of \( x \) but not for others.

**TAYLOR SERIES:** Given a function \( f(x) \) and a center \( c \)
(usually \( c = 0 \)), the Taylor series is \( x \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n
\]

**Main Examples:** (Be able to recognize them!)

1. \( e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \ldots \)

   \[\text{ derivatives of } e^x \text{ at } x=0 = 1, 1, 1, 1, \ldots \]

   \[\text{ Evaluate } \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{1}{2})^n = e^{\frac{1}{2}} = \sqrt{e} \]

2. \( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \ldots \)

   \[\text{alternating series } \to \text{ odd numbers } \& \text{ odd factorials} \]

   \[\text{ derivatives of } \sin x : 0, 1, 0, -1, 0, 1, 0, -1, \ldots \]

3. \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \ldots \)

   \[\text{ derivatives of } \cos x : 1, 0, -1, 0, 1, 0, -1, 0, \ldots \]

   \[\text{alternating series } \to \text{ even numbers } \& \text{ even factorials} \]
\( a) \quad \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 \ldots \)

\( \Rightarrow \) notice that, unlike \( \sin x \), the denominator is NOT a factorial 
\( \frac{1}{2n+1} \neq \frac{1}{(2n)!} \)

\( \Rightarrow \) \( \tan^{-1}(x) \neq \sin(x) \)

\( \Rightarrow \) \{ derivatives: 0, 1, 0, -3, 0, 24, 0 \ldots \} 

\( \text{eg. Evaluate } \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) 2^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \left( \frac{1}{2} \right)^{2n+1}} = \tan^{-1}(1) \)

\( \text{eg. Evaluate } \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) 2^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \left( \frac{1}{2} \right)^{2n}} \)

\( \Rightarrow \) manipulate to reach a familiar form of Taylor Series 
\( = \sqrt{2} \cdot \tan^{-1} \left( \frac{1}{2} \right) \)

\( b) \quad \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \ldots \)

\( \text{Set } x = 1 \)

\( \Rightarrow \) \{ derivatives: 0, 1, -1, 2, -6, 24, \ldots \} 

\( \text{eg. Evaluate } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \left( \frac{1}{x^2} \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \left( \frac{3}{2} \right)^n = \ln \left( \frac{3}{2} \right) \)

\( \star \text{ OPERATIONS ON TAYLOR SERIES:} \)
\( \quad \text{• Substitution} \quad \text{• Differentiation} \quad \text{• Integration} \quad \text{• Addition, Subtraction, Multiplication} \)

\( \Rightarrow \) ways of getting new Taylor series from old ones (often easier than computing \( f^{(n)}(0) \) derivatives by hand)

\( \text{(i) SUBSTITUTION} \)

\( \text{eg. Taylor Series of } e^{x^2} \)

\( e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{Substitute } x^2 \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} = 1 + x^2 + \frac{1}{2} x^4 + \frac{1}{6} x^6 \ldots \)
\[ \sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \]

2) **INTEGRATION**

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n x^n \neq \text{Geometric Series with 1st term 1, } \text{Common ratio}(-x) \]

\[ \ln(1+x) = \int_0^x \left( \sum_{n=0}^{\infty} (-1)^n t^n \right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \]

\[ \Rightarrow \text{Substitute } (x-1) \text{ for } (x) \text{ to get Taylor Series of } \ln x \]

\[ f(x) = \int_0^x e^{t^2} dt \]

Taylor Series:
\[ \int_0^x \left( \sum_{n=0}^{\infty} \frac{1}{n!} t^n \right) dt \]

\[ f(x) = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} x^{2n+1} \]

3) **DIFFERENTIATION** (similar to integration)

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots. \]

\[ \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \left( 1 + x + x^2 + x^3 + \ldots \right) \]

\[ \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=1}^{\infty} n x^{n-1} \]

\[ \text{Taylor Series at } x=0 \text{ of } \frac{1}{(1-x)^2} \]

4) **MULTIPLICATION**

\[ \text{Taylor Series of } x, e^x \]

\[ x \left( \sum_{n=0}^{\infty} \frac{1}{n!} (x)^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1} \]

\[ \text{or } \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n \]}