A SEPARABLE DIFFERENTIAL EQUATIONS

\[ f'(x) = 3x^2 \]

Integrate: \[ \int f'(x) \, dx = \int 3x^2 \, dx \Rightarrow f(x) = x^3 + C \] (General Solution)

Introduces constant! (it is important now)

\[ f'(x) = 3f(x) \Rightarrow f'(x) - 3f(x) = 0 \Rightarrow A - 3 = 0 \Rightarrow A = 3 \]

Linear solution of characteristic equation is \( A = 3 \)

\( \Rightarrow \) ONE solution \( e^{3x} \) because: 1st order, linear

General solution: \( f(x) = Ce^{3x} \)

Another Method:

\[ \frac{f'(x)}{f(x)} = 3 \]

Integrate: \[ \int \frac{f'(x)}{f(x)} \, dx = \int 3 \, dx \Rightarrow \]

\[ \left\{ \begin{array}{l}
  u = f(x) \\
  du = f'(x) \, dx
\end{array} \right. \quad \int \frac{du}{u} = 3x + C \Rightarrow \]

\[ \ln |u| + D = 3x + C \Rightarrow \]

\[ \ln |f(x)| = 3x + C - D \Rightarrow \text{doesn't add anything new since both } C, D \text{ are arbitrary} \]

\[ \ln |f(x)| = 3x + C \Rightarrow \]

\[ |f(x)| = e^{3x + C} \Rightarrow \]

\[ f(x) = \pm (e^{3x} \cdot e^C) \Rightarrow \]

\[ f(x) = C_e^{3x} \text{ any arbitrary constant} \]
\[ f'(t) = -\frac{t}{f(t)} \]

- **Separate**: \[ f(t) \cdot f'(t) = -t \] (i.e., move \( f(t) \) to the left)

- **Integrate**: \[ \int f(t) \cdot f'(t) \, dt = -\int t \, dt \]
  \[
  \begin{aligned}
  u &= f(t) \\
  du &= f'(t) \, dt
  \end{aligned}
  \]
  \[
  \frac{1}{2} f(t)^2 + C = -\frac{1}{2} t^2 + C
  \]

- **Solve**: \[ f(t)^2 = -t^2 + C \Rightarrow f(t) = \pm \sqrt{C - t^2} \]

  or simply \( C \)

  (it is just as arbitrary)

**I.V.P.:**
\[ \begin{aligned}
  f'(t) &= -\frac{t}{f(t)} \\
  f(0) &= 3
  \end{aligned} \]

- **Solve**: \[ 3 = f(0) = \pm \sqrt{C - 0^2} \Rightarrow \pm \sqrt{C} = 3 \]

  Initial data tells that the "\( \pm \)" sign is actually a "+" sign, and that \( C = 9 \)

  Therefore \[ f(t) = \sqrt{9 - t^2} \]

**I.V.P.:**
\[ f(0) = -3 \] then \( f(t) = -\sqrt{9 - t^2} \)
Differential Notation:

For the same problem \( \frac{df}{dt} = -\frac{t}{f} \)

\[
\begin{array}{c}
\text{df for } f'(t) \\
\frac{df}{dt}
\end{array}
\quad\begin{array}{c}
f for f(t)
\end{array}
\]

Separate \( df \) and \( dt \) as well

\[
\int df = -\int t dt \implies \int f df = -\int t dt \implies
\]

\[
\frac{1}{2} f^2 = -\frac{1}{2} t^2 + C \quad \text{(then solve as before)}
\]

eg. \( \frac{dy}{dx} = \frac{2y}{x} \)

Separate + Integrate: \( \int \frac{1}{y} dy = \int \frac{2}{x} dx \implies \)

\[
\ln |y| = 2\ln |x| + C \implies
\]

\[
\ln |y| = \ln (x^2 \cdot e^C) \implies
\]

\[
|y| = e^C \cdot x^2 \implies
\]

Solve for \( y \):

\[
y = \pm e^C \cdot x^2
\]

\( y = C \cdot x^2 \quad \text{(General Solution)} \)

e.g. Newton’s Law of Cooling

\( T(s) = \text{temperature of a metal rod after } s \text{ seconds} \)

Law of Cooling: \( T'(s) = k \cdot (A - T(s)) \)

\( \text{ambient temperature} \quad \text{of the room} \)
\( \text{dependant on} \quad \text{the rod} \)
\[ \text{Rate of cooling/heating is directly proportional to the} \]
\[ \text{temperature difference} \]

- Separate: \[ \frac{T'(s)}{A - T(s)} = K \]

- Integrate: \[ \int \frac{T'(s)}{A - T(s)} \, ds = \int K \, ds \Rightarrow \begin{cases} u = T(s) \\ du = T'(s) \, ds \end{cases} \]

\[ \Rightarrow -\ln|A - T(s)| = ks + C \Rightarrow \]

\[ \Rightarrow \ln|A - T(s)| = -ks - C \Rightarrow \]

\[ \Rightarrow A - T(s) = \pm e^{-ks} - C \Rightarrow \]

\[ \Rightarrow A - T(s) = (\pm e^{-c}) \cdot e^{-ks} \Rightarrow \]

\[ T(s) = A - C \cdot e^{-ks} \Rightarrow \]

ambient \( T \) dependant on type of rod

dependant on

initial temperature